2 Solving Systems of Equations and Inequalities

2.1 Solving Linear Systems Using Substitution
2.2 Solving Linear Systems Using Elimination
2.3 Solving Linear Systems Using Technology
2.4 Solving Systems of Linear Inequalities

Mathematical Thinking: Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.
Maintaining Mathematical Proficiency

Rewriting Literal Equations (A.12.E)

Example 1  Solve the literal equation $3x - 9y = 15$ for $x$.

$$3x - 9y = 15$$ 
Write the equation.

$$3x - 9y + 9y = 15 + 9y$$
Add $9y$ to each side.

$$3x = 15 + 9y$$
Simplify.

$$\frac{3x}{3} = \frac{15 + 9y}{3}$$
Divide each side by $3$.

$$x = 5 + 3y$$
Simplify.

The rewritten literal equation is $x = 5 + 3y$.

Solve the literal equation for $y$.
1. $4y - 4x = 16$
2. $3y + 12x = 18$
3. $2x - 10 = 4y + 6$
4. $x = 7y - y$
5. $x = 4y + 3y + 6$
6. $2y + 6x = z$

Graphing Linear Inequalities in Two Variables (A.3.D)

Example 2  Graph $x + 2y > 8$ in a coordinate plane.

Step 1  Graph $x + 2y = 8$, or $y = -\frac{1}{2}x + 4$.
Use a dashed line because the inequality symbol is $>$. 

Step 2  Test $(0, 0)$.

$$x + 2y > 8$$
Write the inequality.

$$0 + 2(0) > 8$$
Substitute.

$$0 \not> 8$$
Simplify.

Step 3  Because $(0, 0)$ is not a solution, shade the half-plane that does not contain $(0, 0)$.

Graph the inequality in a coordinate plane.

7. $x - 2y < 0$
8. $2x + 2y > 3$
9. $3x + 5y \geq 8$
10. $-x - 6y \leq 12$
11. $x > -4$
12. $y \leq 7$

13. **ABSTRACT REASONING**  Can you always use $(0, 0)$ as a test point when graphing a linear inequality in two variables? Explain your reasoning.
Mathematical Thinking

Mathematically proficient students select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems. (2A.1.C)

Using a Graphing Calculator

Core Concept

Graphing a System of Linear Inequalities

You can use a graphing calculator to find all solutions, if they exist, of a system of linear inequalities.

1. Enter the inequalities into a graphing calculator.
2. Graph the inequalities in an appropriate viewing window, so that the intersection of the half-planes is visible.
3. Find the intersection of the half-planes, which is the graph of all the solutions of the system.

Example 1  Using a Graphing Calculator

Use a graphing calculator to find the solution, if it exists, of the system of linear inequalities.

\[
\begin{align*}
y & \geq 2x - 1 & \text{Inequality 1} \\
y & < \frac{1}{2}x + 5 & \text{Inequality 2}
\end{align*}
\]

Solution

The slopes of the boundary lines are not the same, so you know that the lines intersect. Enter the inequalities into a graphing calculator. Then graph the inequalities in an appropriate viewing window.

Find the intersection of the half-planes. Note that any point on the boundary line \( y = 2x - 1 \) is a solution, and any point on the boundary line \( y = \frac{1}{2}x + 5 \) is not a solution. One solution is \((-1, 2)\).

Monitoring Progress

Use a graphing calculator to graph the system of linear inequalities. Name one solution, if any, of the system.

1. \( y \leq 3x - 2 \quad y > -x + 4 \)
2. \( x + y > -3 \quad -6x + y < 1 \)
3. \( 2x - \frac{1}{2}y \geq 4 \quad 4x - y \leq 5 \)
Essential Question  How can you determine the number of solutions of a linear system?

A linear system is consistent when it has at least one solution. A linear system is inconsistent when it has no solution.

Recognizing Graphs of Linear Systems

Work with a partner. Match each linear system with its corresponding graph. Explain your reasoning. Then classify the system as consistent or inconsistent.

a.  \[2x - 3y = 3\]
   \[-4x + 6y = 6\]

b.  \[2x - 3y = 3\]
   \[x + 2y = 5\]

c.  \[2x - 3y = 3\]
   \[-4x + 6y = -6\]

Solving Systems of Linear Equations

Work with a partner. Solve each linear system by substitution. Then use the graph of the system below to check your solution.

a.  \[2x + y = 5\]
   \[x - y = 1\]

b.  \[x + 3y = 1\]
   \[-x + 2y = 4\]

c.  \[x + y = 0\]
   \[3x + 2y = 1\]

Communicate Your Answer

3. How can you determine the number of solutions of a linear system?

4. Suppose you were given a system of three linear equations in three variables. Explain how you would solve such a system by substitution.

5. Apply your strategy in Question 4 to solve the linear system.

\[x + y + z = 1\]  
Equation 1

\[x - y - z = 3\]  
Equation 2

\[-x - y + z = -1\]  
Equation 3
What You Will Learn

- Visualize solutions of systems of linear equations in three variables.
- Solve systems of linear equations in three variables by substitution.
- Solve real-life problems.

Visualizing Solutions of Systems

A linear equation in three variables is an equation of the form \( ax + by + cz = d \), where \( a, b, \) and \( c \) are not all zero.

The following is an example of a system of three linear equations in three variables.

\[
\begin{align*}
3x + 4y - 8z &= -3 \\
x + y + 5z &= -12 \\
4x - 2y + z &= 10 \\
\end{align*}
\]

A solution of such a system is an ordered triple \((x, y, z)\) whose coordinates make each equation true.

The graph of a linear equation in three variables is a plane in three-dimensional space. The graphs of three such equations that form a system are three planes whose intersection determines the number of solutions of the system, as shown in the diagrams below.

**Exactly One Solution**

The planes intersect in a single point, which is the solution of the system.

**Infinitely Many Solutions**

The planes intersect in a line. Every point on the line is a solution of the system.

The planes could also be the same plane. Every point in the plane is a solution of the system.

**No Solution**

There are no points in common with all three planes.
Solving a Three-Variable System by Substitution

**Core Concept**

**Solving a Three-Variable System by Substitution**

**Step 1** Solve one equation for one of its variables.

**Step 2** Substitute the expression from Step 1 in the other two equations to obtain a linear system in two variables.

**Step 3** Solve the new linear system for both of its variables.

**Step 4** Substitute the values found in Step 3 into one of the original equations and solve for the remaining variable.

When you obtain a false equation, such as $0 = 1$, in any of the steps, the system has no solution.

When you do not obtain a false equation, but obtain an identity such as $0 = 0$, the system has infinitely many solutions.

### EXAMPLE 1   Solving a Three-Variable System (One Solution)

Solve the system by substitution.

\[
\begin{align*}
3y - 6z &= -6 &\text{Equation 1} \\
x - y + 4z &= 10 &\text{Equation 2} \\
2x + 2y - z &= 12 &\text{Equation 3}
\end{align*}
\]

**SOLUTION**

**Step 1** Solve Equation 1 for $y$.

\[y = 2z - 2\]

New Equation 1

**Step 2** Substitute $2z - 2$ for $y$ in Equations 2 and 3 to obtain a system in two variables.

\[
\begin{align*}
x - (2z - 2) + 4z &= 10 &\text{Substitute } 2z - 2 \text{ for } y \text{ in Equation 2.} \\
x + 2z &= 8 &\text{New Equation 2} \\
2x + 2(2z - 2) - z &= 12 &\text{Substitute } 2z - 2 \text{ for } y \text{ in Equation 3.} \\
2x + 3z &= 16 &\text{New Equation 3}
\end{align*}
\]

**Step 3** Solve the new linear system for both of its variables.

\[
\begin{align*}
x &= 8 - 2z &\text{Solve new Equation 2 for } x. \\
2(8 - 2z) + 3z &= 16 &\text{Substitute } 8 - 2z \text{ for } x \text{ in new Equation 3.} \\
z &= 0 &\text{Solve for } z. \\
x &= 8 &\text{Substitute into new Equation 3 to find } x.
\end{align*}
\]

**Step 4** Substitute $x = 8$ and $z = 0$ into an original equation and solve for $y$.

\[
\begin{align*}
3y - 6z &= -6 &\text{Write original Equation 1.} \\
3y - 6(0) &= -6 &\text{Substitute } 0 \text{ for } z. \\
y &= -2 &\text{Solve for } y.
\end{align*}
\]

The solution is $x = 8, y = -2, z = 0$, or the ordered triple $(8, -2, 0)$. Check this solution in each of the original equations.
**Example 2**  
Solving a Three-Variable System (No Solution)

Solve the system by substitution.  
\[
\begin{align*}
4x - y + z &= 5 & \text{Equation 1} \\
8x - 2y + 2z &= 1 & \text{Equation 2} \\
x + y + 7z &= -3 & \text{Equation 3}
\end{align*}
\]

**SOLUTION**

**Step 1**  
Solve Equation 1 for \( z \).  
\[ z = -4x + y + 5 \]  
New Equation 1

**Step 2**  
Substitute \(-4x + y + 5\) for \( z \) in Equations 2 and 3 to obtain a system in two variables.
\[
\begin{align*}
8x - 2y + 2(-4x + y + 5) &= 1 & \text{Substitute } -4x + y + 5 \text{ for } z \text{ in Equation 2.} \\
10 &= 1 & \text{New Equation 2}
\end{align*}
\]

Because you obtain a false equation, you can conclude that the original system has no solution.

**Example 3**  
Solving a Three-Variable System (Many Solutions)

Solve the system by substitution.  
\[
\begin{align*}
4x + y - z &= 2 & \text{Equation 1} \\
4x + y + z &= 2 & \text{Equation 2} \\
12x + 3y - 3z &= 6 & \text{Equation 3}
\end{align*}
\]

**SOLUTION**

**Step 1**  
Solve Equation 1 for \( y \).  
\[ y = -4x + z + 2 \]  
New Equation 1

**Step 2**  
Substitute \(-4x + z + 2\) for \( y \) in Equations 2 and 3 to obtain a system in two variables.
\[
\begin{align*}
4x + (-4x + z + 2) + z &= 2 & \text{Substitute } -4x + z + 2 \text{ for } y \text{ in Equation 2.} \\
z &= 0 & \text{New Equation 2} \\
12x + 3(-4x + z + 2) - 3z &= 6 & \text{Substitute } -4x + z + 2 \text{ for } y \text{ in Equation 3.} \\
6 &= 6 & \text{New Equation 3}
\end{align*}
\]

Because you obtain the identity \( 6 = 6 \), the system has infinitely many solutions.

**Step 3**  
Describe the solutions of the system using an ordered triple. One way to do this is to substitute 0 for \( z \) in Equation 1 to obtain \( y = -4x + 2 \).

So, any ordered triple of the form \((x, -4x + 2, 0)\) is a solution of the system.

**Monitoring Progress**  
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Solve the system by substitution. Check your solution, if possible.

1. \(-x + y + 2z = 7\)  
   \[ x + 3y - z = 5 \]  
   \[ x - 5y + z = -3 \]

2. \(x - y + 2z = 4\)  
   \[ x - y - 2z = 4 \]  
   \[-3x + 3y + 2z = -12 \]

3. \(x + y - 6z = 11\)  
   \[ -2x - 2y + 12z = 18 \]

4. In Example 3, describe the solutions of the system using an ordered triple in terms of \( y \).
Solving Real-Life Problems

**EXAMPLE 4 Applying Mathematics**

An amphitheater charges $75 for each seat in Section A, $55 for each seat in Section B, and $30 for each lawn seat. There are three times as many seats in Section B as in Section A. The revenue from selling all 23,000 seats is $870,000. How many seats are in each section of the amphitheater?

**SOLUTION**

**Step 1** Write a verbal model for the situation.

<table>
<thead>
<tr>
<th>Number of seats in B, $y$</th>
<th>3 \times Number of seats in A, $x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of seats in A, $x$</td>
<td>+ Number of seats in B, $y$ + Number of lawn seats, $z$ = Total number of seats</td>
</tr>
<tr>
<td>75 \times Number of seats in A, $x$</td>
<td>+ 55 \times Number of seats in B, $y$ + 30 \times Number of lawn seats, $z$ = Total revenue</td>
</tr>
</tbody>
</table>

**Step 2** Write a system of equations.

- $y = 3x$ Equation 1
- $x + y + z = 23,000$ Equation 2
- $75x + 55y + 30z = 870,000$ Equation 3

**Step 3** Substitute $3x$ for $y$ in Equations 2 and 3 to obtain a system in two variables.

- $x + 3x + z = 23,000$ Substitute $3x$ for $y$ in Equation 2.
- $4x + z = 23,000$ New Equation 2
- $75x + 55(3x) + 30z = 870,000$ Substitute $3x$ for $y$ in Equation 3.
- $240x + 30z = 870,000$ New Equation 3

**Step 4** Solve the new linear system for both of its variables.

- $z = -4x + 23,000$ Solve new Equation 2 for $z$.
- $240x + 30(-4x + 23,000) = 870,000$ Substitute $-4x + 23,000$ for $z$ in new Equation 3.
- $x = 1500$ Solve for $x$.
- $y = 4500$ Substitute into Equation 1 to find $y$.
- $z = 17,000$ Substitute into Equation 2 to find $z$.

The solution is $x = 1500$, $y = 4500$, and $z = 17,000$, or (1500, 4500, 17,000). So, there are 1500 seats in Section A, 4500 seats in Section B, and 17,000 lawn seats.

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5. **WHAT IF?** On the first day, 10,000 tickets sold, generating $356,000 in revenue. The number of seats sold in Sections A and B are the same. How many lawn seats are still available?
Vocabulary and Core Concept Check

1. **VOCABULARY** The solution of a system of three linear equations is expressed as a(n)_______.

2. **DIFFERENT WORDS, SAME QUESTION** Consider the system of linear equations shown. Which is different? Find “both” answers.

   - $x + 3y = 1$
   - $-x + y + z = 3$
   - $x + 3y - 2z = -7$

   - Solve the system of linear equations.
   - Solve each equation in the system for $y$.
   - Find the ordered triple whose coordinates make each equation true.
   - Find the point of intersection of the planes modeled by the linear system.

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, determine whether the ordered triple is a solution of the system. Justify your answer.

3. $(4, -5, 1)$

   - $2x + y + 5z = 8$
   - $x + 3y + 2z = -9$
   - $-x - 2y + z = -13$

4. $(-2, 3, -6)$

   - $-x + 2y + 2z = -4$
   - $4x + y - 3z = 13$
   - $x - 5y + z = -23$

In Exercises 5–14, solve the system by substitution. (See Example 1.)

5. $x = 4$
   - $x + y = -6$
   - $4x - 3y + 2z = 26$

6. $2x - 3y + z = 10$
   - $y + 2z = 13$
   - $z = 5$

7. $x + 2y = -1$
   - $-x + 3y + 2z = -4$
   - $-x + y - 4z = 10$

8. $2x - 2y + z = 3$
   - $5y - z = -31$
   - $x + 3y + 2z = -21$

9. $12x + 6y + 7z = -35$
   - $7x - 5y - 6z = 200$
   - $x + y = -10$

10. $2x + y + z = 12$
    - $5x + 5y + 5z = 20$
    - $x - 4y + z = -21$

11. $x + y + z = 24$
    - $5x + 3y + z = 56$
    - $x + y - z = 0$

12. $-3x + y + 2z = -13$
    - $7x + 2y - 6z = 37$
    - $x - y + 3z = -14$

13. $-3x - 4y + z = -16$
    - $x + 11y - 2z = 30$
    - $-9x - 4y - z = -4$

14. $x - 3y + 6z = 21$
    - $3x + 2y - 5z = -30$
    - $2x - 5y + 2z = -6$

**ERROR ANALYSIS** In Exercises 15 and 16, describe and correct the error in the first steps of solving the system of linear equations.

15. $z = 11 - 3x - 2y$
   - $x - y - 11 - 3x - 2y = -2$
   - $-2x - 3y = 9$

16. $y = -2 - x + z$
   - $2x + (-2 - x + z) - 2z = 23$
   - $x - z = 25$

In Exercises 17–22, solve the system by substitution. (See Examples 2 and 3.)

17. $y + 3z = 3$
    - $x + 2y + z = 8$
    - $2x + 3y - z = 1$

18. $x = y - z$
    - $x + y + 2z = 1$
    - $3x + 3y + 6z = 4$

19. $2x + y - 3z = -2$
    - $7x + 3y - z = 11$
    - $-4x - 2y + 6z = 4$

20. $11x + 11y - 11z = 44$
    - $22x - 30y + 15z = -8$
    - $x + y - z = 4$

21. $2x + 3y - z = 6$
    - $3x - 12y + 6z = 9$
    - $-x + 4y - 2z = -3$

22. $x - 3y + z = 2$
    - $2x + y + z = 6$
    - $3x - 9y + 3z = 10$
23. **MODELING WITH MATHEMATICS** A wholesale store advertises that for $20 you can buy one pound each of peanuts, cashews, and almonds. Cashews cost as much as peanuts and almonds combined. You purchase 2 pounds of peanuts, 1 pound of cashews, and 3 pounds of almonds for $36. What is the price per pound of each type of nut? *(See Example 4.)*

24. **MODELING WITH MATHEMATICS** Each year, votes are cast for the rookie of the year in a softball league. The voting results for the top three finishers are shown in the table below. How many points is each vote worth?

<table>
<thead>
<tr>
<th>Player</th>
<th>1st place</th>
<th>2nd place</th>
<th>3rd place</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>23</td>
<td>5</td>
<td>1</td>
<td>131</td>
</tr>
<tr>
<td>Player 2</td>
<td>5</td>
<td>17</td>
<td>4</td>
<td>80</td>
</tr>
<tr>
<td>Player 3</td>
<td>1</td>
<td>5</td>
<td>15</td>
<td>35</td>
</tr>
</tbody>
</table>

25. **WRITING** Write a linear system in three variables for which it is easier to solve for one variable than to solve for either of the other two variables. Explain your reasoning.

26. **REPEATED REASONING** Using what you know about solving linear systems in two and three variables by substitution, plan a strategy for how you would solve a system that has four linear equations in four variables.

27. **PROBLEM SOLVING** The number of left-handed people in the world is one-tenth the number of right-handed people. The percent of right-handed people is nine times the percent of left-handed people and ambidextrous people combined. What percent of people are ambidextrous?

28. **MODELING WITH MATHEMATICS** Use a system of linear equations to model the data in the following newspaper article. Solve the system to find how many athletes finished in each place.

**Lawrence High prevailed in Saturday’s track meet with the help of 20 individual-event placers earning a combined 68 points. A first-place finish earns 5 points, a second-place finish earns 3 points, and a third-place finish earns 1 point. Lawrence had a strong second-place showing, with as many second place finishers as first- and third-place finishers combined.**

**MATHEMATICAL CONNECTIONS** In Exercises 29 and 30, write and use a linear system to answer the question.

29. The triangle has a perimeter of 65 feet. What are the lengths of sides \( \ell \), \( m \), and \( n \)?

\[
\ell = \frac{1}{3}m \\
n = \ell + m - 15
\]

30. What are the measures of angles \( A \), \( B \), and \( C \)?

31. **OPEN-ENDED** Write a system of three linear equations in three variables that has the ordered triple \((-4, 1, 2)\) as its only solution. Justify your answer using the substitution method.

32. **MAKING AN ARGUMENT** A linear system in three variables has no solution. Your friend concludes that it is not possible for two of the three equations to have any points in common. Is your friend correct? Explain your reasoning.
33. **PROBLEM SOLVING** A contractor is hired to build an apartment complex. Each 840-square-foot unit has a bedroom, kitchen, and bathroom. The bedroom will be the same size as the kitchen. The owner orders 980 square feet of tile to completely cover the floors of two kitchens and two bathrooms. Determine how many square feet of carpet is needed for each bedroom.

34. **THOUGHT PROVOKING** Consider the system shown.

\[\begin{align*}
  x - 3y + z &= 6 \\
x + 4y - 2z &= 9 
\end{align*}\]

a. How many solutions does the system have?

b. Make a conjecture about the minimum number of equations that a linear system in \(n\) variables can have when there is exactly one solution.

35. **PROBLEM SOLVING** A florist must make 5 identical bridesmaid bouquets for a wedding. The budget is $160, and each bouquet must have 12 flowers. Roses cost $2.50 each, lilies cost $4 each, and irises cost $2 each. The florist wants twice as many roses as the other two types of flowers combined.

a. Write a system of equations to represent this situation, assuming the florist plans to use the maximum budget.

b. Solve the system to find how many of each type of flower should be in each bouquet.

c. Suppose there is no limitation on the total cost of the bouquets. Does the problem still have exactly one solution? If so, find the solution. If not, give three possible solutions.

36. **HOW DO YOU SEE IT?** Determine whether the system of equations that represents the circles has no solution, one solution, or infinitely many solutions. Explain your reasoning.

\[\begin{align*}
a. & \quad y \\
b. & \quad x 
\end{align*}\]

37. **REASONING** Consider a system of three linear equations in three variables. Describe the possible number of solutions in each situation.

a. The graphs of two of the equations in the system are parallel planes.

b. The graphs of two of the equations in the system intersect in a line.

c. The graphs of two of the equations in the system are the same plane.

38. **ANALYZING RELATIONSHIPS** Use the integers −3, 0, and 1 to write a linear system that has a solution of (30, 20, 17).

\[\begin{align*}
x - 3y + 3z &= 21 \\
___x + ___y + ___z &= -30 \\
2x - 5y + 2z &= -6
\end{align*}\]

39. **ABSTRACT REASONING** Write a linear system to represent the first three pictures below. Use the system to determine how many tangerines are required to balance the apple in the fourth picture. Note: The first picture shows that one tangerine and one apple balance one grapefruit.

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**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Solve the system of linear equations by elimination. *(Skills Review Handbook)*

40. \[\begin{align*}
x + 3y &= 6 \\
-x - 2y &= -5
\end{align*}\]

42. \[\begin{align*}
4x + 2y &= -4 \\
-2x + 6y &= 44
\end{align*}\]

41. \[\begin{align*}
2x - y &= -3 \\
-5x + y &= 3
\end{align*}\]

43. \[\begin{align*}
4x - 3y &= 9 \\
5x - 21y &= -6
\end{align*}\]
Essential Question  How can you rewrite a linear system so that it can be solved using mental math?

A linear system in row-echelon form has a “stair-step” pattern with leading coefficients of 1. A system can be written in row-echelon form by producing a series of equivalent systems. Equivalent systems have the same solution.

EXPLORATION 1  Recognizing Graphs of Linear Systems

Work with a partner. Match each linear system in row-echelon form with its corresponding graph. Explain your reasoning.

a. \( x + 2y = 4 \)
   \( y = 2 \)

   [Graph A]

b. \( x - y = 1 \)
   \( y = 2 \)

   [Graph B]

c. \( x + \frac{1}{2}y = 0 \)
   \( y = 2 \)

   [Graph C]

EXPLORATION 2  Writing Linear Systems in Row-Echelon Form

Work with a partner. Match each linear system with its equivalent system in row-echelon form. Justify your answers.

a. \( y = 1 \)
   \( x - 3y = -2 \)

   A. \( x + 2y = 5 \)
   \( y = 2 \)

b. \( x + 2y = 5 \)
   \( -x - y = -3 \)

   B. \( x + 2y = 7 \)
   \( y = 3 \)

c. \( 2x + 4y = 14 \)
   \( -3x - 5y = -18 \)

   C. \( x - 3y = -2 \)
   \( y = 1 \)

Communicate Your Answer

3. How can you rewrite a linear system so that it can be solved using mental math?

4. Equivalent systems are produced using row operations. Describe the row operations you used in Exploration 2 to produce equivalent systems.

5. Use row operations to write the linear system in three variables in row-echelon form.

\[
\begin{align*}
2x - 2y + 4z &= 6 \\
-x + 2y + z &= 0 \\
-y - 2z &= -2
\end{align*}
\]

Equation 1

Equation 2

Equation 3
What You Will Learn

- Solve systems of linear equations in three variables by elimination.
- Solve systems of linear equations by Gaussian elimination.

Solving Systems of Equations by Elimination

The elimination method for solving systems of linear equations in two variables can also be extended to solve a system of linear equations in three variables.

Core Concept

Solving a Three-Variable System by Elimination

Step 1 Eliminate one variable to obtain a linear system in two variables.

Step 2 Solve the new linear system for both of its variables.

Step 3 Substitute the values found in Step 2 into one of the original equations and solve for the remaining variable.

When you obtain a false equation, such as 0 = 1, in any of the steps, the system has no solution.

When you do not obtain a false equation, but obtain an identity such as 0 = 0, the system has infinitely many solutions.

EXAMPLE 1 Solving a Three-Variable System (One Solution)

Solve the system by elimination.  

\[4x + 2y + 3z = 12 \quad \text{Equation 1}\]
\[16x + 5z = -6 \quad \text{Equation 2}\]
\[6x - y + 4z = -3 \quad \text{Equation 3}\]

SOLUTION

Step 1 Rewrite the system as a linear system in two variables.

\[
\begin{align*}
4x + 2y + 3z &= 12 \\
12x - 2y + 8z &= -6 \\
16x + 11z &= 6
\end{align*}
\]

Add 2 times Equation 3 to Equation 1 (to eliminate \(y\)).

New Equation 1:

\[16x + 11z = 6\]

Step 2 Solve the new linear system for both of its variables.

\[
\begin{align*}
16x + 11z &= 6 \\
-16x - 5z &= 6
\end{align*}
\]

Add \(-1\) times Equation 2 to new Equation 1 (to eliminate \(x\)).

\[6z = 12\]

\[z = 2\]

\[x = -1\]

Solve for \(z\).

Step 3 Substitute \(x = -1\) and \(z = 2\) into an original equation and solve for \(y\).

\[6(-1) - y + 4(2) = -3\]

Write original Equation 3.

\[6(-1) - y + 4(2) = -3\]

Substitute \(-1\) for \(x\) and 2 for \(z\).

\[y = 5\]

Solve for \(y\).

The solution is \(x = -1\), \(y = 5\), and \(z = 2\), or the ordered triple \((-1, 5, 2)\).

Check this solution in each of the original equations.
**Section 2.2  Solving Linear Systems Using Elimination**

### Example 2  Solving a Three-Variable System (No Solution)

Solve the system by elimination.

\[
\begin{align*}
5x + y + z &= 2 \\
5x + 5y + 5z &= 3 \\
4x + y - 3z &= -6
\end{align*}
\]

**Solution**

**Step 1** Rewrite the system as a linear system in two variables.

\[
\begin{align*}
-5x - 5y - 5z &= -10 \\
5x + 5y + 5z &= 3
\end{align*}
\]

Add \(-5\) times Equation 1 to Equation 2.

\[
0 = -7
\]

Because you obtain a false equation, the original system has no solution.

### Example 3  Solving a Three-Variable System (Many Solutions)

Solve the system by elimination.

\[
\begin{align*}
x - y + z &= -3 \\
x - y - z &= -3 \\
5x - 5y + z &= -15
\end{align*}
\]

**Solution**

**Step 1** Rewrite the system as a linear system in two variables.

\[
\begin{align*}
x - y + z &= -3 \\
x - y - z &= -3 \\
2x - 2y &= -6
\end{align*}
\]

Add Equation 1 to Equation 2 (to eliminate \(z\)).

\[
2x - 2y = -6
\]

**Step 2** Solve the new linear system for both of its variables.

\[
\begin{align*}
6x - 6y &= -18
\end{align*}
\]

Add \(-3\) times new Equation 2 to new Equation 3.

\[
0 = 0
\]

Because you obtain the identity \(0 = 0\), the system has infinitely many solutions.

**Step 3** Describe the solutions of the system using an ordered triple. One way to do this is to solve new Equation 2 for \(y\) to obtain \(y = x + 3\). Then substitute \(x + 3\) for \(y\) in original Equation 1 to obtain \(z = 0\).

\[
\begin{align*}
x + 3 + z &= 0 \\
x + 3 + 0 &= 0
\end{align*}
\]

So, any ordered triple of the form \((x, x + 3, 0)\) is a solution of the system.

### Monitoring Progress

Solve the system by elimination. Check your solution, if possible.

1. \(x - 2y + z = -11\)  
2. \(x + y - z = -1\)  
3. \(x + y + z = 8\)  
4. \(3x + 2y - z = 7\)  
5. \(4x + 4y - 4z = -2\)  
6. \(-x + 2y + 4z = -9\)  
7. \(x + 2y + z = 0\)  
8. \(2x + y + 2z = 16\)

4. In Example 3, describe the solutions of the system using an ordered triple in terms of \(y\).
Solving Systems by Gaussian Elimination

Systems written in row-echelon form can be easily solved using substitution. A system in row-echelon form has a “stair-step” pattern with leading coefficients of 1. To solve a system that is not in row-echelon form, use the operations shown below to rewrite the system in its equivalent row-echelon form. This process is called **Gaussian elimination**, after German mathematician Carl Friedrich Gauss (1777–1855).

**Core Concept**

Operations that Produce Equivalent Systems

Each of the following row operations on a system of linear equations produces an equivalent system of linear equations.

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one of the equations to another equation to replace the latter equation.

**EXAMPLE 4 Using Gaussian Elimination to Solve a System**

Solve the system by Gaussian elimination.

\[
\begin{align*}
x - 2y + 3z &= 9 & \text{Equation 1} \\
-x + 3y &= -4 & \text{Equation 2} \\
-5y - 14z &= -23 & \text{Equation 3}
\end{align*}
\]

**SOLUTION**

Because the leading coefficient of the first equation is 1, begin by keeping the \(x\) at the upper left position and eliminating the other \(x\)-term from the first column.

\[
\begin{align*}
x - 2y + 3z &= 9 & \text{Write Equation 1.} \\
-x + 3y &= -4 & \text{Write Equation 2.} \\
 y + 3z &= 5 & \text{Add Equation 1 to Equation 2.}
\end{align*}
\]

\[
\begin{align*}
x - 2y + 3z &= 9 & \text{Equation 1} \\
 y + 3z &= 5 & \text{Equation 2} \\
-5y - 14z &= -23 & \text{Equation 3}
\end{align*}
\]

Adding the first equation to the second equation produces a new second equation.

\[
\begin{align*}
x - 2y + 3z &= 9 & \text{Write Equation 1.} \\
 y + 3z &= 5 & \text{Write Equation 2.} \\
-5y - 14z &= -23 & \text{Add new Equation 2 to Equation 3.}
\end{align*}
\]

Adding 5 times the second equation to the third equation produces a new third equation.

\[
\begin{align*}
x - 2y + 3z &= 9 & \text{Equation 1} \\
y + 3z &= 5 & \text{Equation 2} \\
z &= 2 & \text{Add 5 times Equation 2 to Equation 3.}
\end{align*}
\]

Using substitution, you can conclude that the solution is \(x = 1\), \(y = -1\), and \(z = 2\), or the ordered triple \((1, -1, 2)\).

**Monitoring Progress**

5. Use Gaussian elimination to solve the system of linear equations in Monitoring Progress Question 1.
1. **WRITING** How is solving a linear system by elimination similar to solving a linear system by substitution?

2. **WRITING** Explain how you know when a linear system in three variables has infinitely many solutions.

### Monitoring Progress and Modeling with Mathematics

#### Exercises 3–8

In Exercises 3–8, solve the system by elimination. (See Examples 1, 2, and 3.)

<table>
<thead>
<tr>
<th>Exercise</th>
<th>System of Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( x + y - 2z = 5 ) ( -x + 2y + z = 2 ) ( 2x + 3y - z = 9 )</td>
</tr>
<tr>
<td>4</td>
<td>( x + 4y - 6z = -1 ) ( 2x - y + 2z = -7 ) ( -x + 2y - 4z = 5 )</td>
</tr>
<tr>
<td>5</td>
<td>( 3x - y + 2z = 4 ) ( 6x - 2y + 4z = -8 ) ( 2x - y + 3z = 10 )</td>
</tr>
<tr>
<td>6</td>
<td>( 5x + y - z = 6 ) ( x + y + z = 2 ) ( 12x + 4y = 10 )</td>
</tr>
<tr>
<td>7</td>
<td>( x + 3y - z = 2 ) ( x + y - z = 0 ) ( 3x + 2y - 3z = -1 )</td>
</tr>
<tr>
<td>8</td>
<td>( -2x - 3y + z = -6 ) ( x + y - z = 5 ) ( 7x + 8y - 6z = 31 )</td>
</tr>
</tbody>
</table>

#### Error Analysis

In Exercises 9 and 10, describe and correct the error in the first step of solving the system of linear equations by elimination.

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>( 4x - y + 2z = -18 ) ( -x + 2y + z = 11 ) ( 3x + 3y - 4z = 44 )</td>
</tr>
<tr>
<td>10</td>
<td>( 12x - 3y + 6z = -18 ) ( 3x + 3y - 4z = 44 ) ( 15x + 2z = 26 )</td>
</tr>
</tbody>
</table>

#### Exercises 11–16

In Exercises 11–16, solve the system by Gaussian elimination. (See Example 4.)

<table>
<thead>
<tr>
<th>Exercise</th>
<th>System of Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>( x + y - z = 4 ) ( 3x + 2y + 4z = 17 ) ( -x + 5y + z = 8 )</td>
</tr>
<tr>
<td>12</td>
<td>( 2x - y - z = 15 ) ( 4x + 5y + 2z = 10 ) ( -x - 4y + 3z = -20 )</td>
</tr>
<tr>
<td>13</td>
<td>( x + 2y - z = 3 ) ( 2x + 4y - 2z = 6 ) ( -x - 2y + z = -6 )</td>
</tr>
<tr>
<td>14</td>
<td>( x + 2y + 3z = 4 ) ( -3x + 2y - z = 12 ) ( -2x - 2y - 4z = -14 )</td>
</tr>
<tr>
<td>15</td>
<td>( x + 2y - z = 3 ) ( -2x - y + z = -1 ) ( 6x - 3y - z = -7 )</td>
</tr>
<tr>
<td>16</td>
<td>( 4x + y + 5z = 5 ) ( 8x + 2y + 10z = 10 ) ( x - y - 2z = -2 )</td>
</tr>
</tbody>
</table>

17. **MODELING WITH MATHEMATICS** Three orders are placed at a pizza shop. Two small pizzas, a liter of soda, and a salad cost $14; one small pizza, a liter of soda, and three salads cost $15; and three small pizzas, a liter of soda, and two salads cost $22. How much does each item cost?

18. **MODELING WITH MATHEMATICS** Sam’s Furniture Store places the following advertisement in the local newspaper. Write a system of equations for the three combinations of furniture. What is the price of each piece of furniture? Explain.
19. **MODELING WITH MATHEMATICS** A play is performed for a crowd of 400 people. Adult tickets cost $22 each, student tickets cost $15 each, and tickets for children cost $13.50 each. The revenue for the concert is $7840. There are 40 more children at the concert than students. How many of each type of ticket are sold?

20. **MODELING WITH MATHEMATICS** A stadium has 10,000 seats, divided into box seats, lower-deck seats, and upper-deck seats. Box seats sell for $10, lower-deck seats sell for $8, and upper-deck seats sell for $5. When all the seats for a game are sold, the total revenue is $70,000. The stadium has four times as many upper-deck seats as box seats. Find the number of lower-deck seats in the stadium.

21. **COMPARING METHODS** Determine whether you would use elimination or Gaussian elimination to solve each system. Explain your reasoning.

   a. \[2x + 2y + 5z = -1\]
   \[2x - y + z = 2\]
   \[2x + 4y - 3z = 14\]

   b. \[3x + 2y - 3z = -2\]
   \[7x - 2y + 5z = -14\]
   \[2x + 4y - 4z = 6\]

22. **HOW DO YOU SEE IT?** Consider the diagram below. Write a system of linear equations in three variables that models the situation, where the variables represent the numbers of each type of coin.

23. **CRITICAL THINKING** Find the values of \(a, b,\) and \(c\) so that the linear system shown has \((-1, 2, -3)\) as its only solution. Explain your reasoning.

   \[x + 2y - 3z = a\]
   \[-x - y + z = b\]
   \[2x + 3y - 2z = c\]

24. **THOUGHT PROVOKING** Does the system of linear equations have more than one solution? Justify your answer.

   \[4x + y + z = 0\]
   \[2x + \frac{1}{2}y - 3z = 0\]
   \[-x - \frac{1}{2}y - z = 0\]

25. **OPEN-ENDED** Consider the system of linear equations below. Choose nonzero values for \(a, b,\) and \(c\) so the system satisfies the given condition. Explain your reasoning.

   \[x + y + z = 2\]
   \[ax + by + cz = 10\]
   \[x - 2y + z = 4\]

   a. The system has no solution.
   b. The system has exactly one solution.
   c. The system has infinitely many solutions.

26. **PROBLEM SOLVING** You spend $24 on 21 pounds of apples. You purchase 2 pounds of golden delicious apples for $1.25 per pound. Red delicious apples cost $1.00 per pound and empire apples cost $1.50 per pound. Write a system of equations in row-echelon form that represents this situation. How many pounds of each type of apple did you purchase?

---

**Maintaining Mathematical Proficiency**

Use a graphing calculator to solve the system of linear equations. *(Skills Review Handbook)*

27. \[3x + y = 7\]
   \[-x + 2y = 1\]

28. \[-4x - 3y = -17\]
   \[2x + 7y = 47\]

29. \[-x + 8y = -37\]
   \[11x + 4y = 39\]
### 2.1–2.2 What Did You Learn?

#### Core Vocabulary
- linear equation in three variables, p. 60
- system of three linear equations, p. 60
- solution of a system of three linear equations, p. 60
- ordered triple, p. 60
- Gaussian elimination, p. 70

#### Core Concepts

**Section 2.1**
Visualizing Solutions of Systems, p. 60

Solving a Three-Variable System by Substitution, p. 61

**Section 2.2**
Solving a Three-Variable System by Elimination, p. 69

Operations that Produce Equivalent Systems, p. 70

#### Mathematical Thinking

1. How did you use the information in the newspaper article in Exercise 28 on page 65 to write a system of three linear equations?

2. Explain the strategy you used to choose the values for \(a\), \(b\), and \(c\) in Exercise 25 on page 72.

#### Study Skills

**Ten Steps for Test Taking**

1. As soon as you get the test, turn it over and write down any formulas, calculations, and rules that you still have trouble remembering.

2. Preview the test and mark the questions you know how to do easily. These are the problems you should do first.

3. As you preview the test, you may have remembered other information. Write this information on the back of the test.

4. Based on how many points each question is worth, decide on a progress schedule. You should always have more than half of the test done before half the time has elapsed.

5. Solve the problems you marked while previewing the test.

6. Skip the problems that you suspect will give you trouble.

7. After solving all the problems that you know how to do easily, go back and reread the problems you skipped.

8. Try your best at the remaining problems. Even if you cannot solve a problem, you may be able to get partial credit for a few correct steps.

9. Review the test, looking for any careless errors you may have made.

10. The test is not a race against the other students. Use all the allowed test time.
1. Determine whether each ordered triple is a solution of the system of equations shown. Explain your reasoning. (Section 2.1)
   a. \((-1, 2, 1)\)
   \(-x + y + z = 4\)  
   Equation 1
   b. \((-3, 0, 5)\)
   \(-3x + 2y + 3z = 10\)  
   Equation 2
   c. \((0, -1, 4)\)
   \(x - y - z = -4\)  
   Equation 3
   d. \((4, 2, 6)\)

Solve the system by substitution. (Section 2.1)
2. 
   \(x + y - 3z = -1\)  
   \(y = z\)  
   \(-x + 2y = 1\)
3. 
   \(x + 4z = 1\)  
   \(x + y + 10z = 10\)  
   \(2x - y + 2z = -5\)
4. 
   \(x - 3y + z = 1\)  
   \(x - 2y - 3z = 2\)  
   \(x + y - z = -1\)

Solve the system by elimination. (Section 2.2)
5. 
   \(2x + 4y - z = 7\)  
   \(2x - 4y + 2z = -6\)  
   \(x + 4y + z = 0\)
6. 
   \(x - 2y + 3z = 9\)  
   \(-x + 3y = -4\)  
   \(2x - 5y + 5z = 17\)
7. 
   \(x + 2y - 7z = -4\)  
   \(2x + y + z = 13\)  
   \(3x + 9y - 36z = -33\)

Solve the system by Gaussian elimination. (Section 2.2)
8. 
   \(x - 11y + 4z = 3\)  
   \(2x + 4y - z = 7\)  
   \(5x - 3y + 2z = 3\)
9. 
   \(3x - 3y + 6z = 6\)  
   \(x + 2y - z = 5\)  
   \(5x - 8y + 13z = 7\)
10. 
    \(x + 2z = 5\)  
    \(3x - y - z = -2\)  
    \(6x - y + 5z = 13\)

11. Contestants participate in a pumpkin carving contest. The table shows the results of the voting for the gold, silver, and bronze medalists. The gold medalist earned 38 points, the silver medalist earned 30 points, and the bronze medalist earned 22 points. How many points is each vote worth? (Sections 2.1 and 2.2)

<table>
<thead>
<tr>
<th>Medal</th>
<th>1st place</th>
<th>2nd place</th>
<th>3rd place</th>
</tr>
</thead>
<tbody>
<tr>
<td>gold</td>
<td>6</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>silver</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>bronze</td>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

12. In a football game, a total of 45 points were scored. During the game, there were 13 scoring plays. These plays were a combination of touchdowns, extra-point kicks, and field goals, which are worth 6 points, 1 point, and 3 points, respectively. The same number of extra-point kicks and touchdowns were scored, and there were six times as many touchdowns as field goals. How many touchdowns, extra-points, and field goals were scored during the game? (Sections 2.1 and 2.2)

13. A small corporation borrowed $800,000 to expand its business. Some of the money was borrowed at each 8%, some at 9%, and some at 10%. The simple interest charged was $67,500 and the amount borrowed at 8% was four times the amount borrowed at 10%. How much money was borrowed at each rate? (Sections 2.1 and 2.2)
Essential Question: How can you represent algebraic expressions using a coefficient matrix?

A matrix is a rectangular arrangement of numbers. The dimensions of a matrix with \(m\) rows and \(n\) columns are \(m \times n\). So, the dimensions of matrix \(A\) are \(2 \times 3\).

\[
A = \begin{bmatrix}
  4 & -1 & 5 \\
  6 & 3 & \\
\end{bmatrix}
\]

2 rows
3 columns

EXPLORATION 1: Writing Coefficient Matrices

Work with a partner. Match each set of algebraic expressions with its coefficient matrix. Explain your reasoning.

Sample

Algebraic expressions: \(2x + y + 6z\)
\(-x + 3z\)

Coefficient matrix:
\[
\begin{bmatrix}
  2 & 1 & 6 \\
-1 & 0 & 3 \\
\end{bmatrix}
\]

a. \(4x + 3y\)
\(5x + y\)

b. \(4x + 3z\)
\(5x + y\)

c. \(4x - 3z\)
\(5x - y\)

D. \(4x + 2y + 3z\)
\(3y - z\)

\(4z\)

A. \[
\begin{bmatrix}
  4 & 0 & -3 \\
  5 & -1 & 0 \\
\end{bmatrix}
\]

B. \[
\begin{bmatrix}
  4 & 2 & 3 \\
  0 & 3 & -1 \\
  0 & 0 & 4 \\
\end{bmatrix}
\]

C. \[
\begin{bmatrix}
  4 & 3 \\
  5 & 1 \\
\end{bmatrix}
\]

EXPLORATION 2: Writing Coefficient Matrices

Work with a partner. Write and enter the coefficient matrix for each set of expressions into a graphing calculator.

a. \(2y - 5z\)
\(x + 11y\)

b. \(5y - z\)
\(2x + 4z\)

\(x + z\)

c. \(-x - 3y\)
\(-x + 2y + 9z\)

Communicate Your Answer

3. How can you represent algebraic expressions using a coefficient matrix?

4. Write the algebraic expressions that are represented by the coefficient matrix.

\[
\text{MATRIEXA}3 \ 4 \times 4 \\
\begin{bmatrix}
  1 & 0 & -1 & 6 \\
  3 & 5 & 0 & -8 \\
  4 & 3 & 0 & 0 \\
-9 & 0 & 0 & 3 \\
\end{bmatrix}
\]
What You Will Learn

- Write augmented matrices for systems of linear equations.
- Use technology to solve systems of linear equations in three variables.

Writing Augmented Matrices for Systems

A matrix is a rectangular arrangement of numbers. The dimensions of a matrix with \( m \) rows and \( n \) columns are \( m \times n \) (read “\( m \) by \( n \)”). So, the dimensions of matrix \( A \) are \( 2 \times 3 \). The numbers in the matrix are its elements.

\[
A = \begin{bmatrix}
4 & -1 & 5 \\
0 & 6 & 3
\end{bmatrix}
\]

The element in the first row and third column is 5.

A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the augmented matrix of the system. For example, the system below can be represented by the given augmented matrix.

System: \[
\begin{align*}
5x + 3y - 2z &= 1 \\
x + 2y + 3z &= -1 \\
3x - 4y + z &= 10
\end{align*}
\]

Augmented matrix:

\[
\begin{bmatrix}
5 & 3 & -2 & : & 1 \\
1 & 2 & 3 & : & -1 \\
3 & -4 & 1 & : & 10
\end{bmatrix}
\]

Before you write an augmented matrix, make sure each equation in the system is written in standard form. Include zeros for the coefficients of any missing variables. This determines the order of the constants and coefficients in the augmented matrix.

**EXAMPLE 1** Writing an Augmented Matrix

Write an augmented matrix for the system. Then state the dimensions.

\[
\begin{align*}
y - 2x &= 1 \\
x + 3y &= 12
\end{align*}
\]

**SOLUTION**

Begin by rewriting each equation in the system in standard form.

\[
\begin{align*}
-2x + y &= 1 \\
x + 3y &= 12
\end{align*}
\]

Next, use the coefficients and constants as elements of the augmented matrix.

\[
\begin{bmatrix}
-2 & 1 & : & 1 \\
1 & 3 & : & 12
\end{bmatrix}
\]

The augmented matrix has two rows and three columns, so the dimensions are \( 2 \times 3 \).
EXAMPLE 2 Writing an Augmented Matrix

Write an augmented matrix for the system. Then state the dimensions.

\[3x - 4y = 7 - 2z\]
\[9x + 3y = 3\]
\[2x + 4y - z = 2\]

**SOLUTION**

Begin by rewriting each equation in the system in standard form.

\[3x - 4y + 2z = 7\]
\[9x + 3y + 0z = 3\]
\[2x + 4y - z = 2\]

Next, use the coefficients and constants as elements of the augmented matrix.

\[
\begin{bmatrix}
3 & -4 & 2 & : & 7 \\
9 & 3 & 0 & : & 3 \\
2 & 4 & -1 & : & 2
\end{bmatrix}
\]

The augmented matrix has three rows and four columns, so the dimensions are \(3 \times 4\).

**Monitoring Progress**

Write an augmented matrix for the system. Then state the dimensions.

1. \(x - 8y = -4\)
   \(5x + 2y = 9\)
2. \(x + y = 1 - z\)
   \(7x + 9y - z = -2\)
   \(6x + 4y + 8z = 0\)
3. \(9x - 8y + 3z = 11\)
   \(x - y + 2z = 6\)
   \(-x + 4y = 16\)

**Solving Systems of Equations Using Technology**

Many technology tools have matrix features that you can use to solve a system of linear equations. The augmented matrix in Example 2, rewritten in reduced row-echelon form, is shown below. Observe that the solution to this system is \((x, y, z) = (-1.4, 5.2, 16)\). You can verify this solution in the original system.

\[
\begin{bmatrix}
1 & 0 & 0 & : & -1.4 \\
0 & 1 & 0 & : & 5.2 \\
0 & 0 & 1 & : & 16
\end{bmatrix}
\]

**Core Concept**

**Solving a Linear System Using Technology**

Step 1 Write an augmented matrix for the linear system.

Step 2 Enter the augmented matrix into your graphing calculator.

Step 3 Use the reduced row-echelon form feature to rewrite the system.

Step 4 Interpret the result from Step 3 to solve the linear system.
EXAMPLE 3  Solving a System Using Technology

Use a graphing calculator to solve the system.

\[\begin{align*}
    x + y + z &= 2 \\
    2x + 3y + z &= 3 \\
    x - y - 2z &= -6
\end{align*}\]

**SOLUTION**

**Step 1** Write an augmented matrix for the linear system.

\[
\begin{bmatrix}
    1 & 1 & 1 & : & 2 \\
    2 & 3 & 1 & : & 3 \\
    1 & -1 & -2 & : & -6
\end{bmatrix}
\]

**Step 2** Enter the dimensions and elements of the augmented matrix into your graphing calculator.

\[
\begin{align*}
(A) & = \begin{bmatrix}
    1 & 1 & 1 & 2 \\
    2 & 3 & 1 & 3 \\
    1 & -1 & -2 & -6
\end{bmatrix} \\
\text{ NAMES } \text{ MATH } \text{ EDIT } & \text{ MATRIX } [A] 3 \times 4 \\
\text{ NAMES } \text{ MATH } \text{ EDIT } & \text{ ref } [A]
\end{align*}
\]

**Step 3** Use the *reduced row-echelon form* feature to rewrite the system.

\[
\begin{align*}
(A) & = \begin{bmatrix}
    1 & 0 & 0 & -1 \\
    0 & 1 & 0 & 1 \\
    0 & 0 & 1 & 2
\end{bmatrix}
\end{align*}
\]

**Step 4** Converting the matrix back to a system of linear equations, you have:

\[
\begin{align*}
    1x &= -1 \\
    1y &= 1 \\
    1z &= 2
\end{align*}
\]

The solution is \(x = -1, y = 1, \text{ and } z = 2\), or the ordered triple \((-1, 1, 2)\). Check this solution in each of the original equations.

**Check**

\[
\begin{align*}
    x + y + z &= 2 \quad 2x + 3y + z &= 3 \quad x - y - 2z &= -6 \\
    -1 + 1 + 2 &= 2 \quad 2(-1) + 3(1) + 2 &= 3 \quad -1 - 1 - 2(-2) &= -6 \\
    2 &= 2 \quad 3 &= 3 \quad -6 &= -6
\end{align*}
\]

**Monitoring Progress**

Use a graphing calculator to solve the system.

**4.** \(x + 2y - 3z = -2\) \(x - y + z = -1\) \(3x + 4y - 4z = 4\) 
**5.** \(x + 3y - z = 1\) 
**6.** \(3x + 2y - 5z = -10\) \(6x - z = 8\) \(-y + 3z = -2\)
1. **COMPLETING THE SENTENCE** A matrix derived from a system of linear equations is called the ________ matrix of the system.

2. **WRITING** Describe how to find the solution of a system of linear equations in three variables using technology.

### Monitoring Progress and Modeling with Mathematics

**In Exercises 3–10, write an augmented matrix for the system. Then state the dimensions.**

(See Examples 1 and 2.)

3. \[\begin{align*}3x - 4y &= 7 \\
9x + 11y &= 10\end{align*}\]

4. \[\begin{align*}5y - 4x &= -7 \\
3x - 7y &= 15\end{align*}\]

5. \[\begin{align*}x + 8y - 7z &= 12 \\
5x + 9y + 5z &= 15 \\
6z - 3y - 8x &= 1\end{align*}\]

6. \[\begin{align*}4x - 5y + 2z &= -3 \\
6x + 4y + 9z &= 8 \\
11x - 2y - z &= 7\end{align*}\]

7. \[\begin{align*}x - y + z &= 14 \\
6x - 5z &= 13 \\
-3x + 7y + 8z &= -5\end{align*}\]

8. \[\begin{align*}5x + 2z &= 9 \\
3x + 5y - 8z &= 15 \\
4x + 2y + 9z &= 11\end{align*}\]

9. \[\begin{align*}3x + 2y &= z + 7 \\
5x + 4z &= 8y \\
21x + 9y - 13z &= 6\end{align*}\]

10. \[\begin{align*}-x + 3y &= 5z + 9 \\
2x - 4y + 15z &= 3 \\
4y + 3z &= 6x\end{align*}\]

**ERROR ANALYSIS** In Exercises 11 and 12, describe and correct the error in writing an augmented matrix for the system below.

\[\begin{align*}3x - 9y + 5z &= 8 \\
2x + 11y &= 15 \\
6x - 9z + 14y &= 4\end{align*}\]

11. **The augmented matrix is:**

\[\begin{bmatrix}
3 & -9 & 5 & : & 8 \\
2 & 11 & 0 & : & 15 \\
6 & -9 & 14 & : & 4
\end{bmatrix}\]

12. **The augmented matrix is:**

\[\begin{bmatrix}
3 & -9 & 5 & : & 8 \\
2 & 11 & 15 & : & 0 \\
6 & 14 & -9 & : & 4
\end{bmatrix}\]

**In Exercises 13–22, use a graphing calculator to solve the system.** (See Example 3.)

13. \[\begin{align*}4x + y + 6z &= 7 \\
3x + 3y + 2z &= 17 \\
-x - y + z &= -9\end{align*}\]

14. \[\begin{align*}x + 4y - z &= -7 \\
2x - y + 2z &= 15 \\
-3x + y - 3z &= -22\end{align*}\]

15. \[\begin{align*}x + y + z &= 9 \\
x + y + z &= 3 \\
5x - 2z &= -1\end{align*}\]

16. \[\begin{align*}x - 2y + 3z &= -9 \\
2x + 5y + z &= 10 \\
3x - 6y + 9z &= 12\end{align*}\]

17. \[\begin{align*}x + 3z &= 6 \\
-2x + 3y + z &= -11 \\
3x - y + 2z &= 13 \end{align*}\]

18. \[\begin{align*}4x + y + 6z &= 2 \\
2x + 2y + 4z &= 1 \\
-x - y + z &= -5\end{align*}\]

19. \[\begin{align*}x + y + 4z &= 7 \\
2x - 3y - z &= -24 \\
-4x + 2y + 2z &= 8\end{align*}\]

20. \[\begin{align*}x + y + 2z &= 1 \\
x - y + z &= 0 \\
3x + 3y + 6z &= 4\end{align*}\]

21. \[\begin{align*}x + y + 2z &= 10 \\
-x + 2y + z &= 5 \\
-x + 4y + 3z &= 15\end{align*}\]

22. \[\begin{align*}-2x + 4y + z &= 1 \\
3x - 3y - z &= 2 \\
5x - y - z &= 8\end{align*}\]

23. **MODELING WITH MATHEMATICS** A company sells three types of gift baskets. The basic basket has two movie passes and one package of microwave popcorn, and costs $15.50. The medium basket has two movie passes, two packages of microwave popcorn, and one DVD, and costs $37. The super basket has four movie passes, three packages of microwave popcorn, and two DVDs, and costs $72.50.

**a.** Write an augmented matrix to represent the situation.

**b.** Use a graphing calculator to find the cost of each basket item.
24. **MODELING WITH MATHEMATICS** You go shopping at a local department store with your friend and cousin. You buy one pair of jeans, four pairs of shorts, and two shirts for $84. Your friend buys two pairs of jeans, one pair of shorts, and three shirts for $76. Your cousin buys one pair of jeans, two pairs of shorts, and one shirt for $52.

a. Write an augmented matrix to represent the situation.

b. Use a graphing calculator to find the cost of each piece of clothing.

25. **MODELING WITH MATHEMATICS** You have 85 coins in nickels, dimes, and quarters with a combined value of $13.25. There are twice as many quarters as dimes.

a. Write an augmented matrix to represent the situation.

b. Use a graphing calculator to find the number of each type of coin.

26. **HOW DO YOU SEE IT?** Write a system of equations for the augmented matrix below.

\[
\begin{array}{ccc|c}
2 & -1 & 3 & 4 \\
1 & 8 & 0 & 9 \\
-2 & 4 & 6 & 11 \\
\end{array}
\]

27. **MAKING AN ARGUMENT** Your friend states that the number of rows in an augmented matrix of the system will always be the same as the number of variables in the system. Is your friend correct? Explain your reasoning.

28. **USING TOOLS** Use a graphing calculator to solve the system of four linear equations in four variables.

\[
\begin{align*}
2w + 5x - 4y + 6z &= 0 \\
2x + 2y - 7z &= 52 \\
4w + 8x - 7y + 14z &= -25 \\
3w + 6x - 5y + 10z &= -16
\end{align*}
\]

29. **REASONING** Is it possible to write more than one augmented matrix for a system of linear equations? Explain your reasoning.

30. **MATHEMATICAL CONNECTIONS** The sum of the measures of the angles in \(\triangle ABC\) is 180°. The sum of the measures of angle \(B\) and angle \(C\) is twice the measure of angle \(A\). The measure of angle \(B\) is 32° less than the measure of angle \(C\).

a. Write an augmented matrix to represent the situation.

b. Use a graphing calculator to find the measures of the three angles.

31. **ABSTRACT REASONING** Let \(a\), \(b\), and \(c\) be real numbers. Classify the linear system represented by each matrix as consistent or inconsistent. Explain your reasoning.

a. \[
\begin{bmatrix}
1 & 0 & 0 & : & a \\
0 & 1 & 0 & : & b \\
0 & 0 & 1 & : & c
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
1 & 0 & 0 & : & a \\
0 & 1 & 0 & : & b \\
0 & 0 & 0 & : & 0
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
1 & 0 & 0 & : & a \\
0 & 1 & 0 & : & b \\
0 & 0 & 0 & : & 1
\end{bmatrix}
\]

32. **THOUGHT PROVOKING** Write an augmented matrix, not in reduced row-echelon form, for a system that has exactly one solution, \((x, y, z) = (2, 3, 5)\). Justify your answer.

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Solve the inequality. Graph the solution.  
(Skills Review Handbook)

33. \(2 - x > 5\)  
34. \(5z + 8 \geq -7\)

35. \(-2w + 7 \geq 3w + 5\)  
36. \(-\frac{1}{2}r + 1 < \frac{3}{2}r + 7\)

Graph the inequality in a coordinate plane.  
(Skills Review Handbook)

37. \(y < 3x\)  
38. \(y - 4 > 2x + 6\)

39. \(\frac{1}{2}x + y \leq 5\)  
40. \(-x - 3y \geq -9\)
**Essential Question** How can you graph a system of three linear inequalities?

**EXPLORATION 1** Graphing Linear Inequalities

**Work with a partner.** Match each linear inequality with its graph. Explain your reasoning.

- \( x + y \leq 4 \) Inequality 1
- \( x - y \leq 0 \) Inequality 2
- \( y \leq 3 \) Inequality 3

**Graphs:**

A.  

B.  

C.  

**EXPLORATION 2** Graphing a System of Linear Inequalities

**Work with a partner.** Consider the linear inequalities given in Exploration 1.

- \( x + y \leq 4 \) Inequality 1
- \( x - y \leq 0 \) Inequality 2
- \( y \leq 3 \) Inequality 3

a. Use three different colors to graph the inequalities in the same coordinate plane. What is the result?

b. Describe each of the shaded regions of the graph. What does the unshaded region represent?

**Communicate Your Answer**

3. How can you graph a system of three linear inequalities?

4. When graphing a system of three linear inequalities, which region represents the solution of the system?

5. Do you think all systems of three linear inequalities have a solution? Explain your reasoning.

6. Write a system of three linear inequalities represented by the graph.
What You Will Learn

- Check solutions of systems of linear inequalities.
- Graph systems of linear inequalities.
- Write systems of linear inequalities.
- Use systems of linear inequalities to solve real-life problems.

Systems of Linear Inequalities

A **system of linear inequalities** is a set of two or more linear inequalities in the same variables. An example is shown.

A **solution of a system of linear inequalities** is an ordered pair that is a solution of each inequality in the system.

**Example 1** Checking Solutions

Tell whether each ordered pair is a solution of the system of linear inequalities.

\[
\begin{align*}
  y &< 4x & \text{Inequality 1} \\
  y &\geq x - 1 & \text{Inequality 2} \\
  y &\leq 2x + 1 & \text{Inequality 3}
\end{align*}
\]

a. \((1, 0)\)   

**SOLUTION**

a. Substitute 1 for \(x\) and 0 for \(y\) in each inequality.

\[
\begin{array}{ccc}
  \text{Inequality 1} & \text{Inequality 2} & \text{Inequality 3} \\
  y < 4x & y \geq x - 1 & y \leq 2x + 1 \\
  0 < 4(1) & 0 \geq 1 - 1 & 0 \leq 2(1) + 1 \\
  0 < 4 & 0 \geq 0 & 0 \leq 3
\end{array}
\]

Because the ordered pair \((1, 0)\) is a solution of each inequality, it is a solution of the system.

b. Substitute \(-2\) for \(x\) and \(-8\) for \(y\) in each inequality.

\[
\begin{array}{ccc}
  \text{Inequality 1} & \text{Inequality 2} & \text{Inequality 3} \\
  y < 4x & y \geq x - 1 & y \leq 2x + 1 \\
  -8 < 2(-2) & -8 \geq -2 - 1 & -8 \leq 2(-2) + 1 \\
  -8 < -4 & -8 \not\geq -3 & -8 \leq -3
\end{array}
\]

Because \((-2, -8)\) is not a solution of each inequality, it is *not* a solution of the system.

**Monitoring Progress**

Tell whether the ordered pair is a solution of the system of linear inequalities.

1. \((2, 1); y > -x + 3 \quad y \geq x - 4\)
2. \((-2, 3); y \geq 2x - 5 \quad y < x + 6\)
Graphing Systems of Linear Inequalities

The graph of a system of linear inequalities is the graph of all the solutions of the system.

**Core Concept**

**Graphing a System of Linear Inequalities**

**Step 1** Graph each inequality in the same coordinate plane.

**Step 2** Find the intersection of the half-planes that are solutions of the inequalities. This intersection is the graph of the system.

---

**EXAMPLE 2** Graphing a System of Linear Inequalities

Graph the system of linear inequalities.

\[
\begin{align*}
y & > -2x - 5 \\
y & \leq x + 3 \\
y & \geq x - 2
\end{align*}
\]

**SOLUTION**

**Step 1** Graph each inequality.

**Step 2** Find the intersection of the half-planes. One solution is (2, 3).

---

**EXAMPLE 3** Graphing a System of Linear Inequalities

(No Solution)

Graph the system of linear inequalities.

\[
\begin{align*}
2x + 3y & < 6 \\
y & \geq -\frac{2}{3}x + 4 \\
-2x + y & < -2
\end{align*}
\]

**SOLUTION**

**Step 1** Graph each inequality.

**Step 2** Find the intersection of the half-planes. Notice that there is no region shaded red, blue, and green.

So, the system has no solution.
Monitoring Progress

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Graph the system of linear inequalities.

3. \( y < 3x - 2 \)
   \( y > -x + 4 \)
   \( y \leq 2 \)

4. \( x + y > -3 \)

5. \( 2x - \frac{1}{2}y \geq 4 \)
   \( -6x + y < 1 \)
   \( x + y \geq 8 \)

6. \( x \geq -3 \)

7. \( y > \frac{1}{2}x - 2 \)

Writing Systems of Linear Inequalities

EXAMPLE 4 Writing a System of Linear Inequalities

Write a system of linear inequalities represented by the graph.

SOLUTION

Inequality 1  The vertical boundary line passes through \((-3, 0)\). So, an equation of the line is \(x = -3\). The shaded region is to the right of the solid boundary line, so the inequality is \(x \geq -3\).

Inequality 2  The slope of the boundary line is \(\frac{1}{2}\), and the y-intercept is \(-2\). So, an equation of the line is \(y = \frac{1}{2}x - 2\). The shaded region is above the dashed boundary line, so the inequality is \(y > \frac{1}{2}x - 2\).

Inequality 3  The slope of the boundary line is \(-\frac{1}{2}\), and the y-intercept is 1. So, an equation of the line is \(y = -\frac{1}{2}x + 1\). The shaded region is below the dashed boundary line, so the inequality is \(y < -\frac{1}{2}x + 1\).

The system of inequalities represented by the graph is

\[
\begin{align*}
x &\geq -3 & \text{Inequality 1} \\
y &> \frac{1}{2}x - 2 & \text{Inequality 2} \\
y &< -\frac{1}{2}x + 1. & \text{Inequality 3}
\end{align*}
\]

Monitoring Progress

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Write a system of inequalities represented by the graph.

6.

7.
Solving Real-Life Problems

EXAMPLE 5 Applying Mathematics

A discount shoe store is having a sale, as described in the advertisement shown. Use the information in the ad to write and graph a system of inequalities that represents the regular and possible sale prices. How much can regularly priced shoes cost on sale?

SOLUTION

1. Understand the Problem You know the range of regular prices and the range of discounts. You are asked to write and graph a system that represents the situation and determine how much regularly priced shoes cost on sale.

2. Make a Plan Use the given information to write a system of inequalities. Then graph the system and identify an ordered pair in the solution region.

3. Solve the Problem Write a system of inequalities. Let \( x \) be the regular price and let \( y \) be the sale price.

\[
\begin{align*}
  x &\geq 20 & \text{Regular price must be at least } 20. \\
  x &\leq 80 & \text{Regular price must be at most } 80.
\end{align*}
\]

\[
\begin{align*}
  y &\geq 0.4x & \text{Sale price is at least } (100 - 60)\% = 40\% \text{ of regular price.} \\
  y &\leq 0.9x & \text{Sale price is at most } (100 - 10)\% = 90\% \text{ of regular price.}
\end{align*}
\]

Graph the system.

From the graph, you can see that one ordered pair in the solution region is \((50, 30)\). So, a $50 pair of shoes could cost $30 on sale.

4. Look Back Check your solution by substituting \( x = 50 \) and \( y = 30 \) in each inequality in the system, as shown.

\[
\begin{align*}
  x &\geq 20 & \checkmark \\
  50 &\geq 20 & \checkmark \\
  x &\leq 80 & \checkmark \\
  50 &\leq 80 & \checkmark \\
  y &\geq 0.4x & \checkmark \\
  30 &\geq 0.4(50) & \checkmark \\
  30 &\geq 20 & \checkmark \\
  y &\leq 0.9x & \checkmark \\
  30 &\leq 0.9(50) & \checkmark \\
  30 &\leq 45 & \checkmark
\end{align*}
\]

Monitoring Progress

8. Identify and interpret another solution of Example 5.

9. WHAT IF? Suppose all the shoes were sold except those regularly priced from $60 to $80. How does this change the system? Is \((50, 30)\) still a solution? Explain.
1. **VOCABULARY** What must be true in order for an ordered pair to be a solution of a system of linear inequalities?

2. **WHICH ONE DOESN'T BELONG?** Use the graph shown. Which ordered pair does not belong with the other three? Explain your reasoning.

   \((-3, 0)\)  \((-3, 2)\)
   
   \((0, 1)\)  \((-1, 4)\)

---

### In Exercises 3–8, tell whether the ordered pair is a solution of the system of linear inequalities.

(See Example 1.)

3. \((1, 2); y \geq 5x - 6\)  \(y > -x + 2\)  \(y < 3x + 12\)

4. \((8, -2); y \geq -4x - 1\)  \(y < 2x - 2\)

5. \((0, 0); x - y < 4\)  \(y > -\frac{1}{2}x\)  \(x + y \leq 4\)

6. \((-2, 5); y \geq -x\)  \(3x - y > -6\)

7. \((2, 1); x + y > -1\)  \(x - y \geq -2\)  \(x - y \leq 2\)

8. \((1, 3); 2x + y \geq -3\)  \(3x - y > -4\)  \(3x - y \leq 3\)

### In Exercises 9–16, graph the system of linear inequalities.

(See Examples 2 and 3.)

9. \(y < 4\)  \(x > -3\)  \(y > x\)

10. \(y \geq 1\)  \(x \leq 6\)  \(y < 2x - 5\)

11. \(y < 2x + 4\)  \(y \geq 4x - 3\)  \(y < 2x - 3\)

12. \(y < 2x + 1\)  \(y \geq -3x - 1\)  \(y > -3x + 2\)

13. \(2x - 3y > -6\)  \(5x - 3y < 3\)  \(x + 3y > -3\)

14. \(x - 4y > 0\)  \(x + y \leq 1\)  \(x + 3y > -1\)

15. \(y \geq 0\)  \(x \leq 9\)  \(x + y < 15\)

16. \(y < 5\)  \(x \leq 9\)  \(2x + y \geq -1\)

### In Exercises 17–22, write a system of linear inequalities represented by the graph.

(See Example 4.)

17. \(y \geq 0\)  \(x \leq 9\)  \(y < 5\)

18. \(y \leq 9\)  \(x \leq 9\)  \(y < 5\)

19. \(y \geq 0\)  \(x \leq 9\)  \(y < 5\)

20. \(y \leq 9\)  \(x \leq 9\)  \(y < 5\)

21. \(y \geq 0\)  \(x \leq 9\)  \(y < 5\)

22. \(y \leq 9\)  \(x \leq 9\)  \(y < 5\)
ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in graphing the system of linear inequalities.

23. y ≤ 3
   y > 1
   −y ≥ −2x

24. y ≤ 3
   x + y ≥ 5
   x > −1

25. MODELING WITH MATHEMATICS You are buying movie passes and gift cards as prizes for an event. You need at least five movie passes and two gift cards. A movie pass costs $6 and a gift card costs $10. The most you can spend is $70. (See Example 5.)
   a. Write and graph a system of linear inequalities that represents the situation.
   b. Identify and interpret two possible solutions.

26. MODELING WITH MATHEMATICS The Junior-Senior Prom Committee consists of five to eight representatives from the junior and senior classes. The committee must include at least two juniors and at least three seniors.
   a. Write and graph a system of linear inequalities that represents the situation.
   b. Identify and interpret two possible solutions.

27. PROBLEM SOLVING An online media store is having a sale, as described in the advertisement shown. Use the information in the ad to write and graph a system of inequalities for the regular video game prices and possible sale prices. Then use the graph to estimate the range of possible sale prices for games that are regularly priced at $20.

28. PROBLEM SOLVING In baseball, the strike zone is a rectangle the width of home plate that extends from the batter’s knees to a point halfway between the shoulders S and the top T of the uniform pants. The width of home plate is 17 inches. A batter’s knees are 20 inches above the ground and the point halfway between his shoulders and the top of his pants is 42 inches above the ground. Write a system of inequalities that represents the strike zone.

29. MATHEMATICAL CONNECTIONS The following points are the vertices of a rectangle.
   (−3, 3), (4, −2), (−3, −2), (4, 3)
   a. Write a system of linear inequalities that represents the points inside the rectangle.
   b. Find the area of the rectangle.

30. MATHEMATICAL CONNECTIONS The following points are the vertices of a triangle.
   (0, −2), (4, 6), (4, −2)
   a. Write a system of linear inequalities that represents the points inside the triangle.
   b. Find the area of the triangle.

31. USING EQUATIONS Which quadrant of the coordinate plane contains no solutions of the system of linear inequalities?
   y ≥ 4x + 1
   2x + y < 5
   y ≥ −3
   \[ A \] Quadrant I \[ B \] Quadrant II
   \[ C \] Quadrant III \[ D \] Quadrant IV

32. OPEN-ENDED Write a system of three linear inequalities that has (−2, 1) as a solution.
33. **PROBLEM SOLVING** A book on the care of tropical fish states that the pH level of the water should be between 8.0 and 8.3 and the temperature of the water should be between 76°F and 80°F. Let x be the pH level and y be the temperature. Write and graph a system of inequalities that describes the proper pH level and temperature of the water. Compare this graph with the graph you would obtain if the temperatures were given in degrees Celsius.

34. **HOW DO YOU SEE IT?** The graphs of three linear equations are shown.

![Graph showing three linear equations: y = x - 1, y = -2x, and y = -3.]

Replace the equal signs with inequality symbols to create a system of linear inequalities that has point B as a solution, but not point A. Explain your reasoning.

35. **REASONING** Write a system of three linear inequalities for which the solution set consists of the points on the line y = 5x - 2. Justify your answer.

36. **THOUGHT PROVOKING** Is it possible for a system of linear inequalities to have a solution consisting of a single point in a coordinate plane? If so, give an example. If not, explain why.

37. **MAKING AN ARGUMENT** Your friend says that a system of three linear inequalities with three parallel boundary lines has no solution. Is your friend correct? Justify your answer.

38. **MULTIPLE REPRESENTATIONS** A person’s theoretical maximum heart rate y (in beats per minute) is given by $y = 220 - x$, where x is the person’s age in years ($20 \leq x \leq 65$). When a person exercises, it is recommended that the person strive for a heart rate that is at least 50% of the maximum and at most 75% of the maximum.

a. Write a system of linear inequalities that describes the given information.

b. Graph the system you wrote in part (a).

c. A 40-year-old person has a heart rate of 158 beats per minute when exercising. Is the person’s heart rate in the target zone? Explain.

39. **USING TOOLS** Use a graphing calculator to sketch a graph of each system.

a. $y \leq |x|$

b. $y > |2x|$

$y \geq -|x|$

$y < -|2x| + 4$

c. $y \leq |x - 2|$

d. $y \leq |x - 2| + 2$

$y > |x| - 2$

$y \geq |x - 3| - 1$

40. **CRITICAL THINKING** Write a system of linear inequalities that represents the graph of $y > |x - 2|$.

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

**Find the product.** *(Skills Review Handbook)*

41. $(x - 2)^2$

42. $(3m + 1)^2$

43. $(2z - 5)^2$

44. $(4 - y)^2$

**Write a function g described by the given transformation of f(x) = |x| - 5.** *(Section 1.3)*

45. translation 2 units to the left

46. reflection in the x-axis

47. translation 4 units up

48. vertical stretch by a factor of 3
2.3–2.4 What Did You Learn?

Core Vocabulary

- matrix, p. 76
- dimensions of a matrix, p. 76
- elements of a matrix, p. 76
- augmented matrix, p. 76
- system of linear inequalities, p. 82
- solution of a system of inequalities, p. 82
- graph of a system of linear inequalities, p. 83

Core Concepts

Section 2.3
- Writing an Augmented Matrix for a Linear System, p. 76
- Solving a Linear System Using Technology, p. 77

Section 2.4
- Graphing a System of Linear Inequalities, p. 83
- Writing a System of Linear Inequalities, p. 84

Mathematical Thinking

1. In Exercise 15 on page 79, you used a graphing calculator to solve the system of equations. Explain how you could solve this system using paper and pencil.

2. Describe the drawings or diagrams you used to support your claim in Exercise 37 on page 88.

Fun and Games

You are creating a number game for your school newspaper. To pique readers’ interest, you begin with a simple addition problem: Find three numbers whose sum is 20. How do you proceed so that there is only one winner? How do you proceed so that everyone wins? Could you manipulate the game so that no one can possibly win?

To explore the answers to these questions and more, go to BigIdeasMath.com.
### 2.1 Solving Linear Systems Using Substitution  
(pp. 59–66)

**Solve the system by substitution.**

\[
\begin{align*}
2x + 6y + z &= 1 & \text{Equation 1} \\
-x - 3y + 2z &= 7 & \text{Equation 2} \\
x - y - 3z &= -14 & \text{Equation 3}
\end{align*}
\]

**Step 1** Solve Equation 3 for \( x \).

\[
x = y + 3z - 14 \quad \text{New Equation 3}
\]

**Step 2** Substitute \( y + 3z - 14 \) for \( x \) in Equations 1 and 2.

\[
\begin{align*}
2(y + 3z - 14) + 6y + z &= 1 & \text{Substitute } y + 3z - 14 \text{ for } x \text{ in Equation 1.} \\
8y + 7z &= 29 & \text{New Equation 1} \\
-(y + 3z - 14) - 3y + 2z &= 7 & \text{Substitute } y + 3z - 14 \text{ for } x \text{ in Equation 2.} \\
-4y - z &= -7 & \text{New Equation 2}
\end{align*}
\]

**Step 3** Solve the new linear system for both of its variables.

\[
\begin{align*}
z &= -4y + 7 & \text{Solve new Equation 2 for } z. \\
8y + 7(-4y + 7) &= 29 & \text{Substitute } -4y + 7 \text{ for } z \text{ in new Equation 1.} \\
y &= 1 & \text{Solve for } y. \\
z &= 3 & \text{Substitute into new Equation 2 to find } z.
\end{align*}
\]

**Step 4** Substitute \( y = 1 \) and \( z = 3 \) into an original equation and solve for \( x \).

\[
\begin{align*}
x - y - 3z &= -14 & \text{Write original Equation 3.} \\
x - (1) - 3(3) &= -14 & \text{Substitute 1 for } y \text{ and 3 for } z. \\
x &= -4 & \text{Solve for } x.
\end{align*}
\]

*The solution is \( x = -4, \ y = 1, \) and \( z = 3, \) or the ordered triple \((-4, 1, 3).\) Check this solution in each of the original equations.*

**Solve the system by substitution. Check your solution, if possible.**

1. \( \begin{align*}
x + y + z &= 3 \\
-x + 3y + 2z &= -8 \\
x &= 4z
\end{align*} \)

2. \( \begin{align*}
4x + 5y + 3z &= 11 \\
x - 3y + z &= 6 \\
-2x + 6y - 2z &= -12
\end{align*} \)

3. \( \begin{align*}
-2x + y - 5z &= -13 \\
3x + y &= 12 \\
x + y - z &= 2
\end{align*} \)

4. \( \begin{align*}
x - y + z &= 6 \\
-4x + 3y + 2z &= 12 \\
2x - 2y + 2z &= 8
\end{align*} \)

5. \( \begin{align*}
x - y + 3z &= 6 \\
x - 2y &= 5 \\
2x - 2y + 5z &= 9
\end{align*} \)

6. \( \begin{align*}
3x + 2y + z &= 20 \\
-x - y - 2z &= -2 \\
2y + z &= -1
\end{align*} \)

7. A school band performs a spring concert for a crowd of 600 people. The revenue for the concert is $3150. There are 150 more adults at the concert than students. How many of each type of ticket are sold?
2.2 Solving Linear Systems Using Elimination (pp. 67–72)

Solve the system by elimination.

\[
\begin{align*}
x - y + z &= -3 & \text{Equation 1} \\
2x - y + 5z &= 4 & \text{Equation 2} \\
4x + 2y - z &= 2 & \text{Equation 3}
\end{align*}
\]

Step 1 Rewrite the system as a linear system in two variables.

\[
\begin{align*}
x - y + z &= -3 & \text{Add Equation 1 to} \\
4x + 2y - z &= 2 & \text{Equation 3 (to eliminate } z). \\
5x + y &= -1 & \text{New Equation 3} \\
-5x + 5y - 5z &= 15 & \text{Add } -5 \text{ times Equation 1 to} \\
2x - y + 5z &= 4 & \text{Equation 2 (to eliminate } z). \\
-3x + 4y &= 19 & \text{New Equation 2}
\end{align*}
\]

Step 2 Solve the new linear system for both of its variables.

\[
\begin{align*}
-20x - 4y &= 4 & \text{Add } -4 \text{ times new Equation 3 to new Equation 2.} \\
-3x + 4y &= 19 & \\
-23x &= 23 \\
x &= -1 \quad \text{Solve for } x. \\
y &= 4 \quad \text{Substitute into new Equation 2 or 3 to find } y.
\end{align*}
\]

Step 3 Substitute \( x = -1 \) and \( y = 4 \) into an original equation and solve for \( z \).

\[
\begin{align*}
x - y + z &= 23 & \text{Write original Equation 1.} \\
(-1) - 4 + z &= 23 & \text{Substitute } -1 \text{ for } x \text{ and 4 for } y. \\
z &= 2 \quad \text{Solve for } z.
\end{align*}
\]

The solution is \( x = -1, \ y = 4, \text{ and } z = 2, \) or the ordered triple \((-1, 4, 2)\). Check this solution in each of the original equations.

Solve the system by elimination. Check your solution, if possible.

8. \( 2x - 5y - z = 17 \) \( x + y + 3z = 19 \) \( -4x + 6y + z = -20 \)
9. \( x + y + z = 2 \) \( 2x - 3y + z = 11 \) \( -3x + 2y - 2z = -13 \)
10. \( x + 4y - 2z = 3 \) \( x + 3y + 7z = 1 \) \( 2x + 9y - 13z = 2 \)

11. Solve the system below by Gaussian elimination.

\[
\begin{align*}
x - y + 2z &= -8 \\
x + y + z &= 6 \\
3x + 3y + 4z &= 28
\end{align*}
\]

12. A box office sells balcony seats, ground level seats, and VIP passes for shows on tour. The table shows the numbers of each type of ticket sold and the revenues for the first three shows of a tour. What is the price of each type of ticket?

<table>
<thead>
<tr>
<th>Balcony</th>
<th>Ground Level</th>
<th>VIP</th>
<th>Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Show 1</td>
<td>135</td>
<td>280</td>
<td>29</td>
</tr>
<tr>
<td>Show 2</td>
<td>150</td>
<td>270</td>
<td>58</td>
</tr>
<tr>
<td>Show 3</td>
<td>130</td>
<td>265</td>
<td>29</td>
</tr>
</tbody>
</table>
### 2.3 Solving Linear Systems Using Technology (pp. 75–80)

**Use a graphing calculator to solve the system.**

8x - 5y + 13z = -37 \hspace{1cm} \text{Equation 1}

-6x + 14y + 3z = 74 \hspace{1cm} \text{Equation 2}

11x + 4y - 26z = 113 \hspace{1cm} \text{Equation 3}

**Step 1** Write an augmented matrix for the linear system.

\[
\begin{bmatrix}
8 & -5 & 13 & : & -37 \\
-6 & 14 & 3 & : & 74 \\
11 & 4 & -26 & : & 113
\end{bmatrix}
\]

**Step 2** Enter the dimensions and elements of the augmented matrix into your graphing calculator.

**Step 3** Use the reduced row-echelon form feature to rewrite the system.

**Step 4** Converting back to a system of linear equations, you have

1. \( x = 3 \)
2. \( y = 7 \)
3. \( z = -2 \)

The solution to the system is \((3, 7, -2)\).

**Use a graphing calculator to solve the system.**

13. \( x - 3y + z = 2 \)
14. \( 4x - 16y + z = -4 \)
15. \( x - 3y + 2z = -16 \)

-3x + 2y - 2z = 13
4x + y - 3z = 26
2x + y - z = -3

-2x + 6y - 2z = 30
2x - 8y + z = -2
5x + 4y - 3z = 1

### 2.4 Solving Systems of Linear Inequalities (pp. 81–88)

**Graph the system of linear inequalities.**

\( y \geq x + 3 \) \hspace{1cm} \text{Inequality 1}

\( y > -2x + 1 \) \hspace{1cm} \text{Inequality 2}

\( y < -\frac{1}{2}x + 4 \) \hspace{1cm} \text{Inequality 3}

**Step 1** Graph each inequality.

**Step 2** Find the intersection of the half-planes.

One solution is \((-1, 4)\).

**Graph the system of linear inequalities.**

16. \( y > 2x + 1 \)
17. \( 2x + y \geq -2 \)
18. \( y < 2x + 4 \)

\( y < -x + 2 \)
\( x - y < -3 \)
\( -2x + y > 5 \)

\( y \geq 3x + 4 \)
\( 4x - y < 0 \)
\( y \geq 2x + 6 \)

19. You want to work at least 10 hours, but less than 20 hours next week. You earn $8 per hour working at a convenience store and $6 per hour mowing lawns. You need to earn at least $92 to cover your weekly expenses. Write and graph a system of linear inequalities to model this situation.
Chapter 2

Chapter Test

Solve the system using any algebraic method. Check your solution, if possible.

1. \[2x + y - z = 8\]
   \[−4x − y + 2z = −16\]
   \[−2x − 4y − 5z = −26\]

2. \[−2x + z = 6\]
   \[−2x + y + 3z = 14\]
   \[4x − 2z = 3\]

3. \[3x − 4y − z = 6\]
   \[−3x + 6y + 2z = 5\]
   \[−6x + 12y + 4z = 10\]

4. Is it possible for the solution set of a system of linear inequalities to contain all the points in a coordinate plane? Explain.

5. Determine whether each ordered pair is a solution of the system of inequalities shown. Explain your reasoning.
   a. \((3, 1)\)
   b. \((1, 2)\)
   c. \((2, 3)\)
   d. \((3, 0)\)

Use a graphing calculator to solve the system.

6. \[3x − 4y + 2z = −9\]
   \[−2x + 2y + 5z = 16\]
   \[−x + 2y − 7z = −7\]

7. \[2x + y + 2z = 11\]
   \[x + z = 6\]
   \[−3y + z = 7\]

8. \[9x + 27y − 3z = 18\]
   \[−3x − 9y + z = −9\]
   \[6x + 4y + 18z = 7\]

9. The juice bar at a health club receives a delivery of juice at the beginning of each month. Over a three-month period, the health club received 1200 gallons of orange juice, 900 gallons of pineapple juice, and 1000 gallons of grapefruit juice. The table shows the compositions of each juice delivery. How many gallons of juice did the health club receive in each delivery? Justify your answer.

<table>
<thead>
<tr>
<th>Juice</th>
<th>1st delivery</th>
<th>2nd delivery</th>
<th>3rd delivery</th>
</tr>
</thead>
<tbody>
<tr>
<td>orange</td>
<td>70%</td>
<td>50%</td>
<td>30%</td>
</tr>
<tr>
<td>pineapple</td>
<td>20%</td>
<td>30%</td>
<td>30%</td>
</tr>
<tr>
<td>grapefruit</td>
<td>10%</td>
<td>20%</td>
<td>40%</td>
</tr>
</tbody>
</table>

10. Piñatas are made to sell at a craft fair. It takes 2 hours to make a mini piñata and 3 hours to make a regular-sized piñata. The owner of the craft booth must make at least 7 mini piñatas and at least 3 regular-sized piñatas. The craft booth owner has no more than 30 hours available to make piñatas and wants to have at least 10 piñatas to sell.

   a. Write and graph a system of linear inequalities that represents the situation.
   b. Use the graph to determine whether the owner of the craft booth can make 9 mini piñatas and 4 regular-sized piñatas.

11. A produce store sells three different fruit baskets, as described in the advertisement shown. Write an augmented matrix to represent the situation. Then use a graphing calculator to determine the price per pound of each item.
1. Based on the data in the graph, which conclusion is most accurate? (TEKS 2A.8.C)
   - A) The sunspot data show a positive correlation.
   - B) The sunspot data show a negative correlation.
   - C) The sunspot data show approximately no correlation.
   - D) The sunspot data show a strong correlation.

2. Which ordered triple is a solution of the system? (TEKS 2A.3.B)
   - F) (7, 1, −3) \[2x + 5y + 3z = 10\]
   - G) (7, 1, 3) \[3x - y + 4z = 8\]
   - H) (7, −1, −3) \[5x - 2y + 7z = 12\]
   - I) none of the above

3. Which equation produces the narrowest graph? (TEKS 2A.6.C)
   - A) \[y = -1.5|x|\]
   - B) \[y = -\frac{2}{3}|x|\]
   - C) \[y = 0.5|x|\]
   - D) \[y = 2|x|\]

4. GRIDDED ANSWER The table shows the atomic weights of three compounds. Let \(H\), \(N\), and \(O\) represent the atomic weights of hydrogen, nitrogen, and oxygen, respectively. What is the atomic weight of nitrogen? (TEKS 2A.3.B)

<table>
<thead>
<tr>
<th>Compound</th>
<th>Formula</th>
<th>Atomic weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>nitric acid</td>
<td>HNO(_3)</td>
<td>63</td>
</tr>
<tr>
<td>nitrous oxide</td>
<td>N(_2)O</td>
<td>44</td>
</tr>
<tr>
<td>water</td>
<td>H(_2)O</td>
<td>18</td>
</tr>
</tbody>
</table>

5. The indicated weight on a tube of toothpaste is 3.5 ounces. The actual weight varies by up to 0.25 ounce. Which inequality gives the range of actual weights \(w\) (in ounces) of the tube of toothpaste? (TEKS 2A.6.F)
   - F) \[w \geq 3.75\]
   - G) \[w \leq 3.25\]
   - H) \[-0.25 \leq w \leq 0.25\]
   - I) \[3.25 \leq w \leq 3.75\]
6. A specialty shop sells gift baskets that contain three different types of candles: tapers, pillars, and jar candles. Tapers cost $1 each, pillars cost $4 each, and jar candles cost $6 each. Each basket contains 8 candles costing a total of $24, and the number of tapers is equal to the number of pillars and jar candles combined. Which system of equations represents the situation, where \( t \) is the number of tapers, \( p \) is the number of pillars, and \( j \) is the number of jar candles? (TEKS 2A.3.A)

   \[ \begin{align*}
   A & : t + p + j = 24 \\
   & \quad t + 4p + 6j = 8 \\
   & \quad t - p - j = 0 \\
   B & : t + p + j = 8 \\
   & \quad t + 4p + 6j = 24 \\
   & \quad t - p - j = 0 \\
   C & : t + p + j = 0 \\
   & \quad t + 4p + 6j = 24 \\
   & \quad t - p - j = 8 \\
   D & : t - p - j = 8 \\
   & \quad t + p + j = 0
   \end{align*} \]

7. Which equation has the graph shown? (TEKS 2A.6.C)

   \[ \begin{align*}
   F & : y = \frac{3}{2}|x| \\
   G & : y = \frac{2}{3}|x| \\
   H & : y = -\frac{3}{2}|x| \\
   J & : y = -\frac{2}{3}|x|
   \end{align*} \]

8. You and a friend are meeting at a gym. You both agree to meet between 9:00 A.M. and 9:30 A.M. and will wait for each other for up to 10 minutes. The graph represents this situation, where \( x \) is your arrival time and \( y \) is your friend’s arrival time (in minutes after 9:00 A.M.). Which ordered pair represents reasonable arrival times for you and your friend? (TEKS 2A.3.G)

   \[ \begin{align*}
   A & : (15, 10) \\
   B & : (20, 5) \\
   C & : (4, 22) \\
   D & : (29, 10)
   \end{align*} \]

9. Which system of inequalities best represents the shaded region? (TEKS 2A.3.E)

   \[ \begin{align*}
   F & : x > -3 \\
   & \quad x < 4 \\
   & \quad y > -2 \\
   & \quad y < 3 \\
   G & : x \geq -3 \\
   & \quad x \leq 4 \\
   & \quad y \geq -2 \\
   & \quad y \leq 3 \\
   H & : x \leq -3 \\
   & \quad x \geq 4 \\
   & \quad y \leq -2 \\
   & \quad y \geq 3 \\
   J & : x < -3 \\
   & \quad x > 4 \\
   & \quad y < -2 \\
   & \quad y > 3
   \end{align*} \]