Big Ideas Learning is pleased to introduce a new high school program, *Big Ideas Math Integrated Mathematics I, II, and III*. The program was written by renowned authors Ron Larson and Laurie Boswell and was developed using the consistent, dependable learning and instructional theory that have become synonymous with *Big Ideas Math*. Students will gain a deeper understanding of mathematics by narrowing their focus to fewer topics at each grade level. They will also master content through inductive reasoning opportunities, engaging explorations, concise stepped-out examples, and rich thought-provoking exercises.

The *Big Ideas Math Integrated Mathematics* research-based curriculum features a continual development of concepts that have been previously taught while integrating algebra, geometry, probability, and statistics topics throughout each course.

In *Integrated Mathematics I*, students will study linear and exponential equations and functions. Students will use linear regression and perform data analysis. They will also learn about geometry topics such as simple proofs, congruence, and transformations.

*Integrated Mathematics II* expands into quadratic, absolute value, and other functions. Students will also explore polynomial equations and factoring, and probability and its applications. Coverage of geometry topics extends to polygon relationships, proofs, similarity, trigonometry, circles, and three-dimensional figures.

In *Integrated Mathematics III*, students will expand their understanding of area and volume with geometric modeling, which students will apply throughout the course as they learn new types of functions. Students will study polynomial, radical, logarithmic, rational, and trigonometric functions. They will also learn how visual displays and statistics relate to different types of data and probability distributions.
About the Authors

Ron Larson, Ph.D., is well known as the lead author of a comprehensive program for mathematics that spans middle school, high school, and college courses. He holds the distinction of Professor Emeritus from Penn State Erie, The Behrend College, where he taught for nearly 40 years. He received his Ph.D. in mathematics from the University of Colorado. Dr. Larson’s numerous professional activities keep him actively involved in the mathematics education community and allow him to fully understand the needs of students, teachers, supervisors, and administrators.

Laurie Boswell, Ed.D., is the Head of School and a mathematics teacher at the Riverside School in Lyndonville, Vermont. Dr. Boswell is a recipient of the Presidential Award for Excellence in Mathematics Teaching and has taught mathematics to students at all levels, from elementary through college. Dr. Boswell was a Tandy Technology Scholar and served on the NCTM Board of Directors from 2002 to 2005. She currently serves on the board of NCSM and is a popular national speaker.

Dr. Ron Larson and Dr. Laurie Boswell began writing together in 1992. Since that time, they have authored over two dozen textbooks. In their collaboration, Ron is primarily responsible for the student edition while Laurie is primarily responsible for the teaching edition.
# Program Resources

## Print

<table>
<thead>
<tr>
<th>Resource</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Student Edition</strong></td>
<td>Designed to the UDL Guidelines</td>
</tr>
<tr>
<td><strong>Teaching Edition</strong></td>
<td>Laurie’s Notes</td>
</tr>
<tr>
<td><strong>Student Journal</strong></td>
<td>Available in English and Spanish</td>
</tr>
<tr>
<td><strong>Resources by Chapter</strong></td>
<td></td>
</tr>
<tr>
<td>Start Thinking</td>
<td></td>
</tr>
<tr>
<td>Warm Up</td>
<td></td>
</tr>
<tr>
<td>Cumulative Review Warm Up</td>
<td></td>
</tr>
<tr>
<td>Practice A and B</td>
<td></td>
</tr>
<tr>
<td>Enrichment and Extension</td>
<td></td>
</tr>
<tr>
<td>Puzzle Time</td>
<td></td>
</tr>
<tr>
<td>Family Communication Letters</td>
<td>Available in English and Spanish</td>
</tr>
<tr>
<td><strong>Assessment Book</strong></td>
<td></td>
</tr>
<tr>
<td>Performance Tasks</td>
<td></td>
</tr>
<tr>
<td>Prerequisite Skills Test with Item Analysis</td>
<td></td>
</tr>
<tr>
<td>Quarterly Standards Based Tests</td>
<td></td>
</tr>
<tr>
<td>Quizzes</td>
<td></td>
</tr>
<tr>
<td>Chapter Tests</td>
<td></td>
</tr>
<tr>
<td>Alternative Assessments with Scoring Rubrics</td>
<td></td>
</tr>
<tr>
<td>Pre-Course Test with Item Analysis</td>
<td></td>
</tr>
<tr>
<td>Post Course Test with Item Analysis</td>
<td></td>
</tr>
</tbody>
</table>

## Technology

<table>
<thead>
<tr>
<th>Resource</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Student Edition</strong></td>
<td><em>With complete English and Spanish audio</em></td>
</tr>
<tr>
<td></td>
<td>Dynamic eBook App</td>
</tr>
<tr>
<td></td>
<td>Dynamic Solutions Tool</td>
</tr>
<tr>
<td></td>
<td>Dynamic Investigations</td>
</tr>
<tr>
<td></td>
<td>Lesson Tutorial Videos</td>
</tr>
<tr>
<td><strong>Dynamic Classroom</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Vocabulary Flash Cards</td>
</tr>
<tr>
<td></td>
<td>Worked-Out Solutions</td>
</tr>
<tr>
<td></td>
<td>Extra Examples</td>
</tr>
<tr>
<td></td>
<td>Warm Up and Closure Activities</td>
</tr>
<tr>
<td><strong>Dynamic Teaching Tools</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Interactive Whiteboard Lesson Library</em></td>
</tr>
<tr>
<td></td>
<td>• Compatible with SMART®, Promethean®, and Mimio® technology</td>
</tr>
<tr>
<td></td>
<td>Real-Life STEM Videos</td>
</tr>
<tr>
<td></td>
<td><em>Editable Online Resources</em></td>
</tr>
<tr>
<td></td>
<td>• Lesson Plans</td>
</tr>
<tr>
<td></td>
<td>• Assessment Book</td>
</tr>
<tr>
<td></td>
<td>• Resources by Chapter</td>
</tr>
<tr>
<td></td>
<td>• Differentiating the Lesson</td>
</tr>
<tr>
<td></td>
<td>Answer Presentation Tool</td>
</tr>
<tr>
<td><strong>Dynamic Assessment and Progress Monitoring Tool</strong></td>
<td><em>Assessment Creation and Delivery</em></td>
</tr>
<tr>
<td></td>
<td><em>Progress Monitoring</em></td>
</tr>
<tr>
<td><strong>Multilingual Glossary</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Key mathematical vocabulary terms in 14 languages</td>
</tr>
</tbody>
</table>
**Program Overview**

**Program Philosophy: Rigor and Balance with Real-Life Applications**

The *Big Ideas Math®* program balances conceptual understanding with procedural fluency. Real-life applications help turn mathematical learning into an engaging and meaningful way to see and explore the real world.

**Essential Question**

How are the midsegments of a triangle related to the sides of the triangle?

**EXPLORATION 1 Midsegments of a Triangle**

Work with a partner. Use dynamic geometry software. Draw any \( \triangle ABC \).

a. Plot midpoint \( D \) of \( AB \) and midpoint \( E \) of \( BC \). Draw \( DE \), which is a midsegment of \( \triangle ABC \).

**Example 1 Using Midsegments in the Coordinate Plane**

In \( \triangle JKL \), show that midsegment \( MN \) is parallel to \( JL \) and that \( MN = \frac{1}{2} JL \).

**Solution**

Step 1 Find the coordinates of \( M \) and \( N \) by finding the midpoints of \( JK \) and \( KL \).

\[
M = \left( \frac{-6 + (-2)}{2}, \frac{1 + 5}{2} \right) = \left( \frac{-8}{2}, \frac{6}{2} \right) = M(-4, 3)
\]

\[
N = \left( \frac{-2 + 5}{2}, \frac{-1}{2} \right) = \left( \frac{3}{2}, \frac{-1}{2} \right) = N(0.5, 2)
\]

Step 2 Find and compare the slopes of \( MN \) and \( JL \).

\[
slope of MN = \frac{2 - 3}{0 - (-4)} = \frac{-1}{4} \quad slope of JL = \frac{-1 - 1}{2 - (-6)} = \frac{-2}{8} = \frac{-1}{4}
\]

Because the slopes are the same, \( MN \) is parallel to \( JL \).

**Example 3 Using the Triangle Midsegment Theorem**

Triangles are used for strength in roof trusses. In the diagram, \( UV \) and \( VW \) are midsegments of \( \triangle RST \).

Find \( UV \) and \( RS \).

**Solution**

\[
UV = \frac{1}{2} \cdot RT = \frac{1}{2}(90 \text{ in.}) = 45 \text{ in.}
\]

\[
RS = 2 \cdot VW = 2(57 \text{ in.}) = 114 \text{ in.}
\]
Dynamic Technology Package

The *Big Ideas Math* program includes a comprehensive technology package that enhances the curriculum and allows students to engage with the underlying mathematics in the text.

**Dynamic Student Edition**

The **Dynamic Student Edition** gives students access to the complete textbook and robust embedded digital resources. Interactive investigations, direct links to remediation, and additional resources are linked at point-of-use. Students can customize their Dynamic Student Editions through note taking and bookmarking, and it can also be accessed offline after it has been downloaded to a device. Audio support and Lesson Tutorial Videos are also included in English and Spanish.

**Dynamic Investigations**

**Dynamic Investigations** are powered by Desmos® and GeoGebra®. These interactivities expand on the Explorations in the program and allow students to learn mathematics through a hands-on approach. Teachers and students can integrate these investigations into their discovery learning.

**Real-Life STEM Videos**

Every chapter in the program contains a **Real-Life STEM Video** that ties directly to a Performance Task. Students can explore topics like the speed of light, natural disasters, and wind power while applying their knowledge to a comprehensive project or task.

**Dynamic Classroom**

The **Dynamic Classroom** is an online interactive version of the Student Edition that can be used as a lesson presentation tool. Teachers can present their lessons and have point-of-use access to all of the online resources available that supplement every section of the program.

**Dynamic Teaching Tools**

These tools include an **Interactive Whiteboard Lesson Library** that includes customizable lessons and templates for every section in the program. Lessons are compatible with SMART®, Promethean®, and Mimio® whiteboards. The **Answer Presentation Tool** can be used to display worked-out solutions to homework and test problems from *Big Ideas Math* content.

**Dynamic Assessment and Progress Monitoring Tool**

This online tool allows teachers to create tests by standard or by *Big Ideas Math* content. Teachers can assign any exercise from the student textbook, problems from the program’s ancillary pieces, or additional items from the item bank, and students can complete their assignments within the tool’s interface.
Integrated Mathematics II

Student Edition

Chapter 6

Relationships Within Triangles
6 Relationships Within Triangles

6.1 Proving Geometric Relationships
6.2 Perpendicular and Angle Bisectors
6.3 Bisectors of Triangles
6.4 Medians and Altitudes of Triangles
6.5 The Triangle Midsegment Theorem
6.6 Indirect Proof and Inequalities in One Triangle
6.7 Inequalities in Two Triangles
Maintaining Mathematical Proficiency

Writing an Equation of a Perpendicular Line (Math I)

Example 1  Write the equation of a line passing through the point \((-2, 0)\) that is perpendicular to the line \(y = 2x + 8\).

Step 1  Find the slope \(m\) of the perpendicular line. The line \(y = 2x + 8\) has a slope of 2. Use the Slopes of Perpendicular Lines Theorem.

\[
2 \cdot m = -1 \quad \text{The product of the slopes of } \perp \text{ lines is } -1.
\]

\[
m = -\frac{1}{2} \quad \text{Divide each side by 2.}
\]

Step 2  Find the \(y\)-intercept \(b\) by using \(m = -\frac{1}{2}\) and \((x, y) = (-2, 0)\).

\[
y = mx + b \quad \text{Use the slope-intercept form.}
\]

\[
0 = -\frac{1}{2}(-2) + b
\]

\[
-1 = b \quad \text{Substitute for } m, x, \text{ and } y.
\]

\[
m = -\frac{1}{2} \text{ and } b = -1, \text{ an equation of the line is } y = -\frac{1}{2}x - 1.
\]

Write an equation of the line passing through point \(P\) that is perpendicular to the given line.

1. \(P(3, 1), y = \frac{1}{3}x - 5\)  
2. \(P(4, -3), y = -x - 5\)  
3. \(P(-1, -2), y = -4x + 13\)

Writing Compound Inequalities (Math I)

Example 2  Write each sentence as an inequality.

a. A number \(x\) is greater than or equal to \(-1\) and less than 6.

\[
\{ \text{A number } x \text{ is greater than or equal to } -1 \text{ and less than 6.} \}
\]

\[
x \geq -1 \quad \text{and} \quad x < 6
\]

\[
\text{An inequality is } -1 \leq x < 6.
\]

b. A number \(y\) is at most 4 or at least 9.

\[
\{ \text{A number } y \text{ is at most 4 or at least 9.} \}
\]

\[
y \leq 4 \quad \text{or} \quad y \geq 9
\]

\[
\text{An inequality is } y \leq 4 \text{ or } y \geq 9.
\]

Write the sentence as an inequality.

4. A number \(w\) is at least \(-3\) and no more than 8.

5. A number \(m\) is more than 0 and less than 11.

6. A number \(s\) is less than or equal to 5 or greater than 2.

7. A number \(d\) is fewer than 12 or no less than \(-7\).

8. **ABSTRACT REASONING** Is it possible for the solution of a compound inequality to be all real numbers? Explain your reasoning.
Mathematical Practices

Mathematically proficient students use technological tools to explore concepts.

Lines, Rays, and Segments in Triangles

Core Concept

Lines, Rays, and Segments in Triangles

Perpendicular Bisector
Angle Bisector
Median
Altitude
Midsegment

EXAMPLE 1  Drawing a Perpendicular Bisector

Use dynamic geometry software to construct the perpendicular bisector of one of the sides of the triangle with vertices \( A(-1, 2) \), \( B(5, 4) \), and \( C(4, -1) \). Find the lengths of the two segments of the bisected side.

SOLUTION

The two segments of the bisected side have the same length, \( AD = CD = 2.92 \) units.

Monitoring Progress

Refer to the figures at the top of the page to describe each type of line, ray, or segment in a triangle.

1. perpendicular bisector
2. angle bisector
3. median
4. altitude
5. midsegment
6.1 Proving Geometric Relationships

**Essential Question** How can you prove a mathematical statement?

A **proof** is a logical argument that uses deductive reasoning to show that a statement is true.

**EXPLORATION 1** Writing Reasons in a Proof

Work with a partner. Four steps of a proof are shown. Write the reasons for each statement.

**Given** \( AD = AB + AC \)

**Prove** \( CD = AB \)

<table>
<thead>
<tr>
<th>STATEMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( AD = AB + AC )</td>
</tr>
<tr>
<td>2. ( AC + CD = AD )</td>
</tr>
<tr>
<td>3. ( AC + CD = AB + AC )</td>
</tr>
<tr>
<td>4. ( CD = AB )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Given</td>
</tr>
<tr>
<td>2.</td>
</tr>
<tr>
<td>3.</td>
</tr>
<tr>
<td>4.</td>
</tr>
</tbody>
</table>

**EXPLORATION 2** Writing Steps in a Proof

Work with a partner. Five steps of a proof are shown. Complete the statements that correspond to each reason.

**Given** \( m\angle ABD = m\angle CBE \)

**Prove** \( m\angle 1 = m\angle 3 \)

<table>
<thead>
<tr>
<th>STATEMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( m\angle ABD = m\angle 1 + m\angle 2 )</td>
</tr>
<tr>
<td>2. ( m\angle CBE = )</td>
</tr>
<tr>
<td>3.</td>
</tr>
<tr>
<td>4. ( m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3 )</td>
</tr>
<tr>
<td>5.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Angle Addition Postulate</td>
</tr>
<tr>
<td>2. Angle Addition Postulate</td>
</tr>
<tr>
<td>3. Given</td>
</tr>
<tr>
<td>4. Substitution Property of Equality</td>
</tr>
<tr>
<td>5. Subtraction Property of Equality</td>
</tr>
</tbody>
</table>

**Communicate Your Answer**

3. How can you prove a mathematical statement?

4. In Exploration 2, can you prove that \( m\angle 1 = m\angle 2 \)? Explain your reasoning.
6.1 Lesson

What You Will Learn

- Write two-column proofs to prove geometric relationships.
- Write paragraph proofs to prove geometric relationships.
- Write flowchart proofs to prove geometric relationships.
- Write coordinate proofs to prove geometric relationships.

Writing Two-Column Proofs

A proof is a logical argument that uses deductive reasoning to show that a statement is true. There are several formats for proofs. A two-column proof has numbered statements and corresponding reasons that show an argument in a logical order.

In a two-column proof, each statement in the left-hand column is either given information or the result of applying a known property or fact to statements already made. Each reason in the right-hand column is the explanation for the corresponding statement.

EXAMPLE 1 Writing a Two-Column Proof

Write a two-column proof of the Vertical Angles Congruence Theorem.

Given \( \angle 1 \) and \( \angle 3 \) are vertical angles.

Prove \( \angle 1 \cong \angle 3 \)

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle 1 ) and ( \angle 3 ) are vertical angles.</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \angle 1 ) and ( \angle 2 ) are a linear pair. ( \angle 2 ) and ( \angle 3 ) are a linear pair.</td>
<td>2. Definition of linear pair, as shown in the diagram</td>
</tr>
<tr>
<td>3. ( \angle 1 ) and ( \angle 2 ) are supplementary. ( \angle 2 ) and ( \angle 3 ) are supplementary.</td>
<td>3. Linear Pair Postulate</td>
</tr>
<tr>
<td>4. ( \angle 1 \cong \angle 3 )</td>
<td>4. Congruent Supplements Theorem</td>
</tr>
</tbody>
</table>

Monitoring Progress

Help in English and Spanish at BigIdeasMath.com

1. Six steps of a two-column proof are shown. Copy and complete the proof.

Given \( B \) is the midpoint of \( \overline{AC} \).

Prove \( x = 7 \)

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( B ) is the midpoint of ( \overline{AC} ).</td>
<td>1. ( \overline{AB} \cong \overline{BC} )</td>
</tr>
<tr>
<td>2. ( \overline{AB} \cong \overline{BC} )</td>
<td>2. Definition of midpoint</td>
</tr>
<tr>
<td>3. ( AB = BC )</td>
<td>3. Definition of congruent segments</td>
</tr>
<tr>
<td>4. ( 6x = 4x + 14 )</td>
<td>4. ( 2x = 14 )</td>
</tr>
<tr>
<td>5. ( x = 7 )</td>
<td>5. Subtraction Property of Equality</td>
</tr>
<tr>
<td>6. ( x = 7 )</td>
<td>6. ( x = 7 )</td>
</tr>
</tbody>
</table>
Writing Paragraph Proofs

Another proof format is a **paragraph proof**, which presents the statements and reasons of a proof as sentences in a paragraph. It uses words to explain the logical flow of the argument.

**EXAMPLE 2**  Writing a Paragraph Proof

Use the given paragraph proof to write a two-column proof of the Corresponding Angles Theorem.

**Given** \( \overrightarrow{AB} \parallel \overrightarrow{CD} \)

**Prove** \( \angle 1 \cong \angle 2 \)

**Paragraph Proof**

Because \( \overrightarrow{AB} \parallel \overrightarrow{CD} \) and translations map lines to parallel lines, a translation along \( AC \) maps \( AB \) to \( CD \). Because translations are rigid motions, angle measures are preserved, which means the angles formed by \( AB \) and \( AC \) are congruent to the corresponding angles formed by \( CD \) and \( AC \). So, \( \angle 1 \cong \angle 2 \).

**Two-Column Proof**

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overrightarrow{AB} \parallel \overrightarrow{CD} )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. A translation along ( AC ) maps ( AB ) to ( CD ).</td>
<td>2. Translations map lines to parallel lines.</td>
</tr>
<tr>
<td>3. Angles formed by ( AB ) and ( AC ) are congruent to the corresponding angles formed by ( CD ) and ( AC ).</td>
<td>3. Translations are rigid motions.</td>
</tr>
<tr>
<td>4. ( \angle 1 \cong \angle 2 )</td>
<td>4. ( \angle 1 ) and ( \angle 2 ) are corresponding angles.</td>
</tr>
</tbody>
</table>

**Monitoring Progress**

Help in English and Spanish at BigIdeasMath.com

2. Copy and complete the paragraph proof of the Alternate Interior Angles Theorem. Then write a two-column proof.

**Given** \( p \parallel q \)

**Prove** \( \angle 1 \cong \angle 2 \)

\( \angle 1 \cong \angle 3 \) by the _____________________________. \( \angle 3 \cong \angle 2 \) by the Vertical Angles Congruence Theorem. \( \angle 1 \cong \angle 2 \) by the _____________________________.

---

*Copyright © Big Ideas Learning, LLC. All rights reserved.*
Writing Flowchart Proofs

Another proof format is a flowchart proof, or flow proof, which uses boxes and arrows to show the flow of a logical argument. Each reason is below the statement it justifies.

**Example 3  Writing a Flowchart Proof**

Use the given flowchart proof to write a two-column proof of the Triangle Sum Theorem.

**Given** \( \triangle ABC \)

**Prove** \( m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ \)

**Flowchart Proof**

Draw \( \overline{BD} \) parallel to \( \overline{AC} \).

- **Parallel Postulate**
- **Angle Addition Postulate, Definition of straight angle**
- **Substitution Property of Equality**
- **Definition of congruent angles**
- **Alternate Interior Angles Theorem**

**Two-Column Proof**

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Draw ( \overline{BD} ) parallel to ( \overline{AC} ).</td>
<td>1. Parallel Postulate</td>
</tr>
<tr>
<td>2. ( m\angle 4 + m\angle 2 + m\angle 5 = 180^\circ )</td>
<td>2. Angle Addition Postulate and definition of straight angle</td>
</tr>
<tr>
<td>3. ( \angle 1 \cong \angle 4, \angle 3 \cong \angle 5 )</td>
<td>3. Alternate Interior Angles Theorem</td>
</tr>
<tr>
<td>4. ( m\angle 1 = m\angle 4, m\angle 3 = m\angle 5 )</td>
<td>4. Definition of congruent angles</td>
</tr>
<tr>
<td>5. ( m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ )</td>
<td>5. Substitution Property of Equality</td>
</tr>
</tbody>
</table>

**Monitoring Progress**

3. Copy and complete the flowchart proof of the Base Angles Theorem. Then write a two-column proof.
Writing Coordinate Proofs

A coordinate proof involves placing geometric figures in a coordinate plane. When you use variables to represent the coordinates of a figure in a coordinate proof, the results are true for all figures of that type.

**EXAMPLE 4** Writing a Coordinate Proof

Write a coordinate proof.

**Given** $P(0, k), Q(h, 0), R(-h, 0)$

**Prove** $\triangle PQR$ is isosceles.

By the Distance Formula, $PR = \sqrt{h^2 + k^2}$ and $PQ = \sqrt{h^2 + k^2}$. So, $\triangle PQR$ is isosceles by definition.

**Monitoring Progress** Help in English and Spanish at BigIdeasMath.com

4. Write a coordinate proof.

**Given** $T(m, 0), U(0, -m), V(-m, 0), W(0, m)$

**Prove** $\triangle TUV \cong \triangle VWT$

---

**Concept Summary**

**Types of Proofs**

**Symmetric Property of Angle Congruence**

- **Given** $\angle 1 \cong \angle 2$
- **Prove** $\angle 2 \cong \angle 1$

**Two-Column Proof**

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\angle 1 \cong \angle 2$</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. $m\angle 1 = m\angle 2$</td>
<td>2. Definition of congruent angles</td>
</tr>
<tr>
<td>3. $m\angle 2 = m\angle 1$</td>
<td>3. Symmetric Property of Equality</td>
</tr>
<tr>
<td>4. $\angle 2 \cong \angle 1$</td>
<td>4. Definition of congruent angles</td>
</tr>
</tbody>
</table>

**Flowchart Proof**

Given $\angle 1 \cong \angle 2$ → Definition of congruent angles → $m\angle 1 = m\angle 2$ → Symmetric Property of Equality → $m\angle 2 = m\angle 1$ → $\angle 2 \cong \angle 1$ → Definition of congruent angles

**Paragraph Proof**

$\angle 1$ is congruent to $\angle 2$. By the definition of congruent angles, the measure of $\angle 1$ is equal to the measure of $\angle 2$. The measure of $\angle 2$ is equal to the measure of $\angle 1$ by the Symmetric Property of Equality. Then by the definition of congruent angles, $\angle 2$ is congruent to $\angle 1$. 
6.1 Exercises

Vocabulary and Core Concept Check

1. **WRITING** Describe how paragraph proofs and flowchart proofs differ in how they present the reason for each step.

2. **COMPLETE THE SENTENCE** In a two-column proof, the ______ are on the left and the _____ are on the right.

Monitoring Progress and Modeling with Mathematics

**PROOF** In Exercises 3–6, copy and complete the proof. (See Example 1.)

3. **Right Angles Congruence Theorem**
   - **Given** ∠1 and ∠2 are right angles.
   - **Prove** ∠1 ≅ ∠2
   - **STATEMENTS**
     1. ∠1 and ∠2 are right angles.
     2. \( m\angle 1 = 90^\circ \), \( m\angle 2 = 90^\circ \)
     3. 
     4. \( \angle 1 \cong \angle 2 \)
   - **REASONS**
     1. 
     2. Definition of right angle
     3. Transitive Property of Equality
     4. Definition of congruent angles

4. **Statement about congruent angles**
   - **Given** ∠1 ≅ ∠4
   - **Prove** ∠2 ≅ ∠3
   - **STATEMENTS**
     1. \( \angle 1 \cong \angle 4 \)
     2. \( \angle 1 \cong \angle 2, \; \angle 3 \cong \angle 4 \)
     3. 
     4. \( \angle 2 \cong \angle 3 \)
   - **REASONS**
     1. Given
     2. Vertical Angles Congruence Theorem
     3. Transitive Property of Congruence
     4. 

5. **Statement about congruent angles**
   - **Given** ∠1 ≅ ∠3
   - **Prove** ∠2 ≅ ∠4
   - **STATEMENTS**
     1. \( \angle 1 \cong \angle 3 \)
     2. \( \angle 1 \cong \angle 2, \; \angle 3 \cong \angle 4 \)
     3. 
     4. \( \angle 2 \cong \angle 4 \)
   - **REASONS**
     1. Given
     2. 
     3. Transitive Property of Congruence
     4. Transitive Property of Congruence

6. **Alternate Exterior Angles Theorem**
   - **Given** \( p \parallel q \)
   - **Prove** ∠1 ≅ ∠2
   - **STATEMENTS**
     1. \( p \parallel q \)
     2. 
     3. \( \angle 3 \cong \angle 2 \)
     4. \( \angle 1 \cong \angle 2 \)
   - **REASONS**
     1. Given
     2. Corresponding Angles Theorem
     3. 
     4. Transitive Property of Congruence
PROVING A THEOREM  In Exercises 7 and 8, copy and complete the paragraph proof. Then write a two-column proof.  (See Example 2.)

7. Consecutive Interior Angles Theorem
   \[\text{Given} \quad n \parallel p\]
   \[\text{Prove} \quad \angle 1 \text{ and } \angle 2 \text{ are supplementary.}\]

   Line \(n\) and line \(p\) are parallel. By the \[\underline{\text{____________________}}, \quad \angle 1 \cong \angle 3.\]
   By the definition of congruent angles, \(m\angle 1 = m\angle 3.\)
   By the Linear Pair Postulate, \[\underline{\text{____________________}}.\]
   By the definition of supplementary angles, \[m\angle 2 + m\angle 1 = 180^\circ.\]
   By substitution, \(m\angle 2 + m\angle 1 = 180^\circ.\) So, \(\angle 1\) and \(\angle 2\) are supplementary by the definition of supplementary angles.

8. Perpendicular Transversal Theorem
   \[\text{Given} \quad h \parallel k, j \perp h\]
   \[\text{Prove} \quad j \perp k\]

   Line \(h\) and line \(k\) are parallel, and line \(j\) and line \(h\) are perpendicular. By the definition of perpendicular lines, \(m\angle 2 = \underline{\text{____}}.\)
   By the \[\underline{\text{____________________}}, \quad \angle 2 \cong \angle 6.\]
   By the definition of congruent angles, \[\underline{\text{____________________}}.\]
   By the Transitive Property of Equality, \[____ = 90^\circ.\]
   By the \[\underline{\text{____________________}}, j \perp k.\]

PROVING A THEOREM  In Exercises 9 and 10, copy and complete the flowchart proof. Then write a two-column proof.  (See Example 3.)

9. Corresponding Angles Converse
   \[\text{Given} \quad \angle ABD \cong \angle CDE\]
   \[\text{Prove} \quad \overline{AB} \parallel \overline{CD}\]

   \[\angle ABD \cong \angle CDE\]
   Given

   A translation along \(\overrightarrow{BD}\) maps \(\angle ABD\) onto \(\angle CDE.\)
   Translations map lines to parallel lines.

10. Alternate Interior Angles Converse
    \[\text{Given} \quad \angle 4 \cong \angle 5\]
    \[\text{Prove} \quad g \parallel h\]

    \[\angle 4 \cong \angle 5\]
    Given

    Transitive Property of Congruence

    \[\angle 1 \cong \angle 4\]
    Corresponding Angles Converse
In Exercises 11 and 12, write a coordinate proof.  (See Example 4.)

11. **Given** Coordinates of vertices of \( \triangle OBC \) and \( \triangle ODC \)  
**Prove** \( \triangle OBC \) and \( \triangle ODC \) are isosceles triangles.

![Diagram of \( \triangle OBC \) and \( \triangle ODC \) with vertices labeled]

12. **Given** \( X \) is the midpoint of \( \overline{YW} \).  
**Prove** \( \triangle XYZ \cong \triangle XWO \)

![Diagram of \( \triangle XYZ \) and \( \triangle XWO \) with vertices labeled]

13. **ERROR ANALYSIS** In the diagram, \( \overline{AE} \cong \overline{DE} \) and \( \overline{AE} \cong \overline{BE} \). Describe and correct the error in the reasoning.

**Incorrect Reasoning:**  
Because \( \overline{AE} \cong \overline{DE} \) and \( \overline{AE} \cong \overline{BE} \), then \( \overline{DE} \cong \overline{BE} \) by the Symmetric Property of Congruence.

**Correct Reasoning:**  
Because \( \overline{AE} \cong \overline{DE} \) and \( \overline{AE} \cong \overline{BE} \), then \( \overline{DE} \cong \overline{BE} \) by the Transitive Property of Congruence.

14. **HOW DO YOU SEE IT?** Use the figure to write each proof.

   a. Alternate Exterior Angles Converse
   **Given** \( \angle 2 \cong \angle 7 \)
   **Prove** \( m \parallel n \)

   ![Diagram showing alternate exterior angles]

   b. Consecutive Interior Angles Converse
   **Given** \( \angle 3 \) and \( \angle 5 \) are supplementary.
   **Prove** \( m \parallel n \)

   ![Diagram showing consecutive interior angles]

15. **MAKING AN ARGUMENT** Your friend says that there is enough information to prove that \( \angle ABC \) and \( \angle DBE \) are vertical angles. Is your friend correct? Explain your reasoning.

   ![Diagram showing vertical angles]

16. **THOUGHT PROVOKING** Describe convenient ways in which you can place a figure in a coordinate plane to complete a coordinate proof. Then place an equilateral triangle in a coordinate plane and label the vertices.

---

**Maintaining Mathematical Proficiency**  
Reviewing what you learned in previous grades and lessons

- **17.** \( BD \) bisects \( \angle ABC \) such that \( m \angle ABD = (4x - 5)^{\circ} \) and \( m \angle DBC = (3x + 2)^{\circ} \). (Skills Review Handbook)

- **18.** Find \( m \angle ABC \).

- **19.** Point \( M \) is the midpoint of \( \overline{AB} \). Find the length of \( \overline{AB} \). (Skills Review Handbook)

---

342  Chapter 6  Relationships Within Triangles

Copyright © Big Ideas Learning, LLC. All rights reserved.
6.2 Perpendicular and Angle Bisectors

Essential Question: What conjectures can you make about a point on the perpendicular bisector of a segment and a point on the bisector of an angle?

EXPLORATION 1 Points on a Perpendicular Bisector

Work with a partner. Use dynamic geometry software.

a. Draw any segment and label it \( \overline{AB} \). Construct the perpendicular bisector of \( \overline{AB} \).

b. Label a point \( C \) that is on the perpendicular bisector of \( \overline{AB} \) but is not on \( \overline{AB} \).

c. Draw \( \overline{CA} \) and \( \overline{CB} \) and find their lengths. Then move point \( C \) to other locations on the perpendicular bisector and note the lengths of \( \overline{CA} \) and \( \overline{CB} \).

d. Repeat parts (a)–(c) with other segments. Describe any relationship(s) you notice.

EXPLORATION 2 Points on an Angle Bisector

Work with a partner. Use dynamic geometry software.

a. Draw two rays \( \overline{AB} \) and \( \overline{AC} \) to form \( \angle BAC \). Construct the bisector of \( \angle BAC \).

b. Label a point \( D \) on the bisector of \( \angle BAC \).

c. Construct and find the lengths of the perpendicular segments from \( D \) to the sides of \( \angle BAC \). Move point \( D \) along the angle bisector and note how the lengths change.

d. Repeat parts (a)–(c) with other angles. Describe any relationship(s) you notice.

Communicate Your Answer

3. What conjectures can you make about a point on the perpendicular bisector of a segment and a point on the bisector of an angle?

4. In Exploration 2, what is the distance from point \( D \) to \( \overline{AB} \) when the distance from \( D \) to \( \overline{AC} \) is 5 units? Justify your answer.
6.2 Lesson

What You Will Learn

- Use perpendicular bisectors to find measures.
- Use angle bisectors to find measures and distance relationships.
- Write equations for perpendicular bisectors.

Using Perpendicular Bisectors

Previously, you learned that a perpendicular bisector of a line segment is the line that is perpendicular to the segment at its midpoint.

A point is equidistant from two figures when the point is the same distance from each figure.

Core Vocabulary

- equidistant, p. 344
- perpendicular bisector
- angle bisector

STUDY TIP

A perpendicular bisector can be a segment, a ray, a line, or a plane.

Theorems

Perpendicular Bisector Theorem

In a plane, if a point lies on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

If \( \overline{CP} \) is the \( \perp \) bisector of \( \overline{AB} \), then \( CA = CB \).

Proof  p. 344

Converse of the Perpendicular Bisector Theorem

In a plane, if a point is equidistant from the endpoints of a segment, then it lies on the perpendicular bisector of the segment.

If \( DA = DB \), then point \( D \) lies on the \( \perp \) bisector of \( \overline{AB} \).

Proof  Ex. 32, p. 350

PROOF Perpendicular Bisector Theorem

Given  \( \overline{CP} \) is the perpendicular bisector of \( \overline{AB} \).

Prove  \( CA = CB \)

Paragraph Proof  Because \( \overline{CP} \) is the perpendicular bisector of \( \overline{AB} \), \( \overline{CP} \) is perpendicular to \( \overline{AB} \) and point \( P \) is the midpoint of \( \overline{AB} \). By the definition of midpoint, \( AP = BP \), and by the definition of perpendicular lines, \( m\angle CPA = m\angle CPB = 90^\circ \). Then by the definition of segment congruence, \( \overline{AP} \equiv \overline{BP} \), and by the definition of angle congruence, \( \angle CPA \equiv \angle CPB \). By the Reflexive Property of Congruence, \( \overline{CP} \equiv \overline{CP} \). So, \( \triangle CPA \equiv \triangle CPB \) by the SAS Congruence Theorem, and \( CA \equiv CB \) because corresponding parts of congruent triangles are congruent. So, \( CA = CB \) by the definition of segment congruence.
EXAMPLE 1  Using the Perpendicular Bisector Theorems

Find each measure.

a. \( RS \)

From the figure, \( SQ \) is the perpendicular bisector of \( PR \). By the Perpendicular Bisector Theorem, \( PS = RS \).

\[ \text{So, } RS = PS = 6.8. \]

b. \( EG \)

Because \( EH = GH \) and \( HF \perp EG \), \( HF \) is the perpendicular bisector of \( EG \) by the Converse of the Perpendicular Bisector Theorem. By the definition of segment bisector, \( EG = 2GF \).

\[ \text{So, } EG = 2(9.5) = 19. \]

c. \( AD \)

From the figure, \( BD \) is the perpendicular bisector of \( AC \).

\[ \begin{align*}
AD &= CD & \text{Perpendicular Bisector Theorem} \\
5x &= 3x + 14 & \text{Substitute} \\
x &= 7 & \text{Solve for } x.
\end{align*} \]

\[ \text{So, } AD = 5x = 5(7) = 35. \]

EXAMPLE 2  Solving a Real-Life Problem

Is there enough information in the diagram to conclude that point \( N \) lies on the perpendicular bisector of \( KM \)?

SOLUTION

It is given that \( KL \equiv ML \). So, \( LN \) is a segment bisector of \( KM \). You do not know whether \( LN \) is perpendicular to \( KM \) because it is not indicated in the diagram.

\[ \text{So, you cannot conclude that point } N \text{ lies on the perpendicular bisector of } KM. \]

Monitoring Progress  Help in English and Spanish at BigIdeasMath.com

Use the diagram and the given information to find the indicated measure.

1. \( ZX \) is the perpendicular bisector of \( WY \), and \( YZ = 13.75 \). Find \( WZ \).

2. \( ZX \) is the perpendicular bisector of \( WY \), \( WZ = 4n - 13 \), and \( YZ = n + 17 \). Find \( YZ \).

3. Find \( WX \) when \( WZ = 20.5 \), \( WY = 14.8 \), and \( YZ = 20.5 \).
Using Angle Bisectors

Previously, you learned that an angle bisector is a ray that divides an angle into two congruent adjacent angles. You also know that the distance from a point to a line is the length of the perpendicular segment from the point to the line. So, in the figure, \( AD \) is the bisector of \( \angle BAC \), and the distance from point \( D \) to \( AB \) is \( DB \), where \( DB \perp AB \).

\[ \text{Theorems} \]

**Angle Bisector Theorem**

If a point lies on the bisector of an angle, then it is equidistant from the two sides of the angle.

If \( \overline{AD} \) bisects \( \angle BAC \) and \( \overline{DB} \perp \overline{AB} \) and \( \overline{DC} \perp \overline{AC} \), then \( DB = DC \).

*Proof* Ex. 33(a), p. 350

**Converse of the Angle Bisector Theorem**

If a point is in the interior of an angle and is equidistant from the two sides of the angle, then it lies on the bisector of the angle.

If \( \overline{DB} \perp \overline{AB} \) and \( \overline{DC} \perp \overline{AC} \) and \( DB = DC \), then \( \overline{AD} \) bisects \( \angle BAC \).

*Proof* Ex. 33(b), p. 350

\[ \text{EXAMPLE 3 Using the Angle Bisector Theorems} \]

Find each measure.

\[ a. \quad m\angle GFJ \]

Because \( \overline{JG} \perp \overline{FG} \) and \( \overline{JH} \perp \overline{FH} \) and \( JG = JH = 7 \), \( \overline{FJ} \) bisects \( \angle GFH \) by the Converse of the Angle Bisector Theorem.

\[ \Rightarrow \text{So, } m\angle GFJ = m\angle HFJ = 42^\circ. \]

\[ b. \quad RS \]

\[ PS = RS \quad \text{Angle Bisector Theorem} \]

\[ 5x = 6x - 5 \quad \text{Substitute.} \]

\[ 5 = x \quad \text{Solve for } x. \]

\[ \Rightarrow \text{So, } RS = 6x - 5 = 6(5) - 5 = 25. \]

**Monitoring Progress**

Use the diagram and the given information to find the indicated measure.

4. \( \overline{BD} \) bisects \( \angle ABC \), and \( DC = 6.9 \). Find \( DA \).

5. \( \overline{BD} \) bisects \( \angle ABC \), \( AD = 3z + 7 \), and \( CD = 2z + 11 \). Find \( CD \).

6. Find \( m\angle ABC \) when \( AD = 3.2 \), \( CD = 3.2 \), and \( m\angle DBC = 39^\circ \).
EXAMPLE 4  Solving a Real-Life Problem

A soccer goalie’s position relative to the ball and goalposts forms congruent angles, as shown. Will the goalie have to move farther to block a shot toward the right goalpost \( R \) or the left goalpost \( L \)?

SOLUTION

The congruent angles tell you that the goalie is on the bisector of \( \angle LBR \). By the Angle Bisector Theorem, the goalie is equidistant from \( \overrightarrow{BR} \) and \( \overrightarrow{BL} \).

So, the goalie must move the same distance to block either shot.

Writing Equations for Perpendicular Bisectors

EXAMPLE 5  Writing an Equation for a Bisector

Write an equation of the perpendicular bisector of the segment with endpoints \( P(-2, 3) \) and \( Q(4, 1) \).

SOLUTION

Step 1  Graph \( \overline{PQ} \). By definition, the perpendicular bisector of \( \overline{PQ} \) is perpendicular to \( \overline{PQ} \) at its midpoint.

Step 2  Find the midpoint \( M \) of \( \overline{PQ} \).

\[
M \left( \frac{-2 + 4}{2}, \frac{3 + 1}{2} \right) = M \left( \frac{2}{2}, \frac{4}{2} \right) = M(1, 2)
\]

Step 3  Find the slope of the perpendicular bisector.

\[
\text{slope of } \overline{PQ} = \frac{1 - 3}{4 - (-2)} = \frac{-2}{6} = \frac{-1}{3}
\]

Because the slopes of perpendicular lines are negative reciprocals, the slope of the perpendicular bisector is 3.

Step 4  Write an equation. The bisector of \( \overline{PQ} \) has slope 3 and passes through \((1, 2)\).

\[
y = mx + b \quad \text{Use slope-intercept form.}
\]

\[
2 = 3(1) + b \quad \text{Substitute for } m, x, \text{ and } y.
\]

\[
-1 = b \quad \text{Solve for } b.
\]

So, an equation of the perpendicular bisector of \( \overline{PQ} \) is \( y = 3x - 1 \).

Monitoring Progress

7. Do you have enough information to conclude that \( \overrightarrow{QS} \) bisects \( \angle PQR \)? Explain.

8. Write an equation of the perpendicular bisector of the segment with endpoints \((-1, -5)\) and \((3, -1)\).
Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE  Point $C$ is in the interior of $\angle DEF$. If $\angle DEC$ and $\angle CEF$ are congruent, then $EC$ is the ________ of $\angle DEF$.

2. DIFFERENT WORDS, SAME QUESTION  Which is different? Find “both” answers.

- Is point $B$ the same distance from both $X$ and $Z$?
- Is point $B$ equidistant from $X$ and $Z$?
- Is point $B$ collinear with $X$ and $Z$?
- Is point $B$ on the perpendicular bisector of $XZ$?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the indicated measure. Explain your reasoning. (See Example 1.)

3. $GH$
4. $QR$

5. $AB$
6. $UW$

In Exercises 11–14, find the indicated measure. Explain your reasoning. (See Example 3.)

9. $m\angle ABD$
10. $PS$

11. $m\angle KJL$
12. $FG$

In Exercises 7–10, tell whether the information in the diagram allows you to conclude that point $P$ lies on the perpendicular bisector of $LM$. Explain your reasoning. (See Example 2.)

7. $
8. $
In Exercises 15 and 16, tell whether the information in the diagram allows you to conclude that \(EH\) bisects \(\angle FEG\). Explain your reasoning. (See Example 4.)

15. 

![Diagram](image1)

16. 

![Diagram](image2)

In Exercises 17 and 18, tell whether the information in the diagram allows you to conclude that \(DB = DC\). Explain your reasoning.

17. 

![Diagram](image3)

18. 

![Diagram](image4)

In Exercises 19–22, write an equation of the perpendicular bisector of the segment with the given endpoints. (See Example 5.)

19. \(M(1, 5), N(7, -1)\)

20. \(Q(-2, 0), R(6, 12)\)

21. \(U(-3, 4), V(9, 8)\)

22. \(Y(10, -7), Z(-4, 1)\)

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in the student’s reasoning.

23. 

![Diagram](image5)

24. 

![Diagram](image6)

25. MODELING MATHEMATICS In the photo, the road is perpendicular to the support beam and \(AB \equiv CB\). Which theorem allows you to conclude that \(AD \equiv CD\)?

26. MODELING WITH MATHEMATICS The diagram shows the position of the goalie and the puck during a hockey game. The goalie is at point \(G\), and the puck is at point \(P\).

![Diagram](image7)

a. What should be the relationship between \(PG\) and \(\angle APB\) to give the goalie equal distances to travel on each side of \(PG\)?

b. How does \(m\angle APB\) change as the puck gets closer to the goal? Does this change make it easier or more difficult for the goalie to defend the goal? Explain your reasoning.

27. CONSTRUCTION Use a compass and straightedge to construct a copy of \(XY\). Construct a perpendicular bisector and plot a point \(Z\) on the bisector so that the distance between point \(Z\) and \(XY\) is 3 centimeters. Measure \(XZ\) and \(YZ\). Which theorem does this construction demonstrate?

28. WRITING Explain how the Converse of the Perpendicular Bisector Theorem is related to the construction of a perpendicular bisector.

29. REASONING What is the value of \(x\) in the diagram?

\[\begin{align*}
&\text{A} \quad 13 \\
&\text{B} \quad 18 \\
&\text{C} \quad 33 \\
&\text{D} \quad \text{not enough information}
\end{align*}\]

30. REASONING Which point lies on the perpendicular bisector of the segment with endpoints \(M(7, 5)\) and \(N(-1, 5)\)?

\[\begin{align*}
&\text{A} \quad (2, 0) \\
&\text{B} \quad (3, 9) \\
&\text{C} \quad (4, 1) \\
&\text{D} \quad (1, 3)
\end{align*}\]

31. MAKING AN ARGUMENT Your friend says it is impossible for an angle bisector of a triangle to be the same line as the perpendicular bisector of the opposite side. Is your friend correct? Explain your reasoning.
32. **PROVING A THEOREM** Prove the Converse of the Perpendicular Bisector Theorem. (*Hint: Construct a line through point C perpendicular to AB at point P.*)

![Diagram of triangle ABC with a line through C perpendicular to AB at point P.]

**Given** \( CA = CB \)

**Prove** Point C lies on the perpendicular bisector of \( AB \).

33. **PROVING A THEOREM** Use a congruence theorem to prove each theorem.
   a. Angle Bisector Theorem
   b. Converse of the Angle Bisector Theorem

34. **HOW DO YOU SEE IT?** The figure shows a map of a city. The city is arranged so each block north to south is the same length and each block east to west is the same length.

   ![Map of a city with labeled schools and street names.]

   a. Which school is approximately equidistant from both hospitals? Explain your reasoning.
   b. Is the museum approximately equidistant from Wilson School and Roosevelt School? Explain your reasoning.

35. **MATHEMATICAL CONNECTIONS** Write an equation whose graph consists of all the points in the given quadrants that are equidistant from the x- and y-axes.
   a. I and III
   b. II and IV
   c. I and II

36. **THOUGHT PROVOKING** The postulates and theorems in this book represent Euclidean geometry. In spherical geometry, all points are on the surface of a sphere. A line is a circle on the sphere whose diameter is equal to the diameter of the sphere. In spherical geometry, is it possible for two lines to be perpendicular but not bisect each other? Explain your reasoning.

37. **PROOF** Use the information in the diagram to prove that \( AB \cong CB \) if and only if points \( D, E, \) and \( B \) are collinear.

![Diagram with labeled points and lines]

38. **PROOF** Prove the statements in parts (a)–(c).

   **Given** Plane \( P \) is a perpendicular bisector of \( XZ \) at point \( Y \).

   **Prove**
   a. \( XW \cong ZW \)
   b. \( XV \cong ZV \)
   c. \( \angle VXW \cong \angle VZW \)

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Classify the triangle by its sides. (*Skills Review Handbook*)

39. 

40. 

41. 

Classify the triangle by its angles. (*Skills Review Handbook*)

42. 

43. 

44.
6.3 Bisectors of Triangles

Essential Question  What conjectures can you make about the perpendicular bisectors and the angle bisectors of a triangle?

EXPLORATION 1  Properties of the Perpendicular Bisectors of a Triangle

Work with a partner.  Use dynamic geometry software. Draw any $\triangle ABC$.

a. Construct the perpendicular bisectors of all three sides of $\triangle ABC$. Then drag the vertices to change $\triangle ABC$. What do you notice about the perpendicular bisectors?

b. Label a point $D$ at the intersection of the perpendicular bisectors.

c. Draw the circle with center $D$ through vertex $A$ of $\triangle ABC$. Then drag the vertices to change $\triangle ABC$. What do you notice?

Sample

Points
A(1, 1)  B(2, 4)  C(6, 0)

Segments
$BC = 5.66$  $AC = 5.10$  $AB = 3.16$

Lines
$x + 3y = 9$  $-5x + y = -17$

EXPLORATION 2  Properties of the Angle Bisectors of a Triangle

Work with a partner.  Use dynamic geometry software. Draw any $\triangle ABC$.

a. Construct the angle bisectors of all three angles of $\triangle ABC$. Then drag the vertices to change $\triangle ABC$. What do you notice about the angle bisectors?

b. Label a point $D$ at the intersection of the angle bisectors.

c. Find the distance between $D$ and $AB$. Draw the circle with center $D$ and this distance as a radius. Then drag the vertices to change $\triangle ABC$. What do you notice?

Sample

Points
A(−2, 4)  B(6, 4)  C(5, −2)

Segments
$BC = 6.08$  $AC = 9.22$  $AB = 8$

Lines
$0.35x + 0.94y = 3.06$  $-0.94x - 0.34y = -4.02$

Communicate Your Answer

3. What conjectures can you make about the perpendicular bisectors and the angle bisectors of a triangle?
What You Will Learn

- Use and find the circumcenter of a triangle.
- Use and find the incenter of a triangle.

Using the Circumcenter of a Triangle

When three or more lines, rays, or segments intersect in the same point, they are called **concurrent** lines, rays, or segments. The point of intersection of the lines, rays, or segments is called the **point of concurrency**.

In a triangle, the three perpendicular bisectors are concurrent. The point of concurrency is the **circumcenter** of the triangle.

**Core Vocabulary**

- concurrent, p. 352
- point of concurrency, p. 352
- circumcenter, p. 352
- incenter, p. 355

**Previous**

- perpendicular bisector
- angle bisector

**Theorems**

**Circumcenter Theorem**

The circumcenter of a triangle is equidistant from the vertices of the triangle.

If \( PD, PE, \) and \( PF \) are perpendicular bisectors, then \( PA = PB = PC \).

**Proof** p. 352

**Plan for Proof**

1. \( \triangle ABC \); the perpendicular bisectors of \( AB, BC, \) and \( AC \)
2. The perpendicular bisectors intersect in a point; that point is equidistant from \( A, B, \) and \( C \).
3. Show that \( P \), the point of intersection of the perpendicular bisectors of \( AB \) and \( BC \), also lies on the perpendicular bisector of \( AC \). Then show that point \( P \) is equidistant from the vertices of the triangle.

**STUDY TIP**

Use diagrams like the one below to help visualize your proof.
Three snack carts sell frozen yogurt from points A, B, and C outside a city. Each of the three carts is the same distance from the frozen yogurt distributor.

Find the location of the distributor.

**SOLUTION**

The distributor is equidistant from the three snack carts. The Circumcenter Theorem shows that you can find a point equidistant from three points by using the perpendicular bisectors of the triangle formed by those points.

Copy the positions of points A, B, and C and connect the points to draw \( \triangle ABC \). Then use a ruler and protractor to draw the three perpendicular bisectors of \( \triangle ABC \). The circumcenter \( D \) is the location of the distributor.

**Monitoring Progress**

Help in English and Spanish at BigIdeasMath.com

1. Three snack carts sell hot pretzels from points A, B, and E. What is the location of the pretzel distributor if it is equidistant from the three carts? Sketch the triangle and show the location.

The circumcenter \( P \) is equidistant from the three vertices, so \( P \) is the center of a circle that passes through all three vertices. As shown below, the location of \( P \) depends on the type of triangle. The circle with center \( P \) is said to be circumscribed about the triangle.
CONSTRUCTION  Circumscribing a Circle About a Triangle

Use a compass and straightedge to construct a circle that is circumscribed about \( \triangle ABC \).

\[ \begin{align*}
\text{Step 1} & \quad \text{Draw a bisector} \quad \text{Draw the perpendicular bisector of } AB. \\
\text{Step 2} & \quad \text{Draw a bisector} \quad \text{Draw the perpendicular bisector of } BC. \text{ Label the intersection of the bisectors } D. \text{ This is the circumcenter.} \\
\text{Step 3} & \quad \text{Draw a circle} \quad \text{Place the compass at } D. \text{ Set the width by using any vertex of the triangle. This is the radius of the circumcircle. Draw the circle. It should pass through all three vertices } A, B, \text{ and } C.
\end{align*} \]

STUDY TIP

Note that you only need to find the equations for two perpendicular bisectors. You can use the perpendicular bisector of the third side to verify your result.

MAKING SENSE OF PROBLEMS

Because \( \triangle ABC \) is a right triangle, the circumcenter lies on the triangle.

EXAMPLE 2  Finding the Circumcenter of a Triangle

Find the coordinates of the circumcenter of \( \triangle ABC \) with vertices \( A(0, 3), B(0, -1), \) and \( C(6, -1) \).

\[ \begin{align*}
\text{SOLUTION} \\
\text{Step 1} & \quad \text{Graph } \triangle ABC. \\
\text{Step 2} & \quad \text{Find equations for two perpendicular bisectors. Use the Slopes of Perpendicular Lines Theorem, which states that horizontal lines are perpendicular to vertical lines.} \\
\text{Step 3} & \quad \text{Find the point where } x = 3 \text{ and } y = 1 \text{ intersect. They intersect at } (3, 1). \quad \text{So, the coordinates of the circumcenter are } (3, 1).
\end{align*} \]

Monitoring Progress

Help in English and Spanish at BigIdeasMath.com

Find the coordinates of the circumcenter of the triangle with the given vertices.

2. \( R(-2, 5), S(-6, 5), T(-2, -1) \)
3. \( W(-1, 4), X(1, 4), Y(1, -6) \)
Section 6.3
Bisectors of Triangles

Using the Incenter of a Triangle

Just as a triangle has three perpendicular bisectors, it also has three angle bisectors. The angle bisectors of a triangle are also concurrent. This point of concurrency is the incenter of the triangle. For any triangle, the incenter always lies inside the triangle.

**Theorem**

**Incenter Theorem**

The incenter of a triangle is equidistant from the sides of the triangle.

If \( AP, BP, \) and \( CP \) are angle bisectors of \( \triangle ABC \), then \( PD = PE = PF \).

**Proof** Ex. 38, p. 359

**EXAMPLE 3** Using the Incenter of a Triangle

In the figure shown, \( ND = 5x - 1 \) and \( NE = 2x + 11 \).

a. Find \( NF \).

b. Can \( NG \) be equal to 18? Explain your reasoning.

**SOLUTION**

a. \( N \) is the incenter of \( \triangle ABC \) because it is the point of concurrency of the three angle bisectors. So, by the Incenter Theorem, \( ND = NE = NF \).

Step 1 Solve for \( x \).

\[
\begin{align*}
ND &= NE \\
5x - 1 &= 2x + 11 \\
x &= 4
\end{align*}
\]

Step 2 Find \( ND \) (or \( NE \)).

\[
ND = 5x - 1 = 5(4) - 1 = 19
\]

\( \Rightarrow \) So, because \( ND = NF \), \( NF = 19 \).

b. Recall that the shortest distance between a point and a line is a perpendicular segment. In this case, the perpendicular segment is \( NF \), which has a length of 19. Because \( 18 < 19 \), \( NG \) cannot be equal to 18.

**Monitoring Progress**

4. In the figure shown, \( QM = 3x + 8 \) and \( QN = 7x + 2 \). Find \( QP \).
Because the incenter $P$ is equidistant from the three sides of the triangle, a circle drawn using $P$ as the center and the distance to one side of the triangle as the radius will just touch the other two sides of the triangle. The circle is said to be \textit{inscribed} within the triangle.

**CONSTRUCTION** \textbf{Inscribing a Circle Within a Triangle}

Use a compass and straightedge to construct a circle that is inscribed within $\triangle ABC$.

**SOLUTION**

1. **Step 1**
   - Draw a bisector
   - Draw the angle bisector of $\angle A$.

2. **Step 2**
   - Draw a bisector
   - Draw the angle bisector of $\angle C$. Label the intersection of the bisectors $D$. This is the incenter.

3. **Step 3**
   - Draw a perpendicular line
   - Draw the perpendicular line from $D$ to $AB$. Label the point where it intersects $AB$ as $E$.

4. **Step 4**
   - Draw a circle
   - Place the compass at $D$. Set the width to $E$. This is the radius of the \textit{incircle}. Draw the circle. It should touch each side of the triangle.

**EXAMPLE 4** \textbf{Solving a Real-Life Problem}

A city wants to place a lamppost on the boulevard shown so that the lamppost is the same distance from all three streets. Should the location of the lamppost be at the circumcenter or incenter of the triangular boulevard? Explain.

**SOLUTION**

Because the shape of the boulevard is an obtuse triangle, its circumcenter lies outside the triangle. So, the location of the lamppost cannot be at the circumcenter. The city wants the lamppost to be the same distance from all three streets. By the Incenter Theorem, the incenter of a triangle is equidistant from the sides of a triangle.

- So, the location of the lamppost should be at the incenter of the boulevard.

**Monitoring Progress**

5. Draw a sketch to show the location $L$ of the lamppost in Example 4.
6.3 Exercises

Vocabulary and Core Concept Check

1. **VOCABULARY** When three or more lines, rays, or segments intersect in the same point, they are called ______ lines, rays, or segments.

2. **WHICH ONE DOESN'T BELONG?** Which triangle does not belong with the other three? Explain your reasoning.

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, the perpendicular bisectors of \( \triangle ABC \) intersect at point \( G \) and are shown in blue. Find the indicated measure.

3. Find \( BG \).

4. Find \( GA \).

In Exercises 5 and 6, the angle bisectors of \( \triangle XYZ \) intersect at point \( P \) and are shown in red. Find the indicated measure.

5. Find \( PB \).

6. Find \( HP \).

In Exercises 7–10, find the coordinates of the circumcenter of the triangle with the given vertices. (See Example 2.)

7. \( A(2, 6), B(8, 6), C(8, 10) \)

8. \( D(-7, -1), E(-1, -1), F(-7, -9) \)

9. \( H(-10, 7), J(-6, 3), K(-2, 3) \)

10. \( L(3, -6), M(5, -3), N(8, -6) \)

In Exercises 11–14, \( N \) is the incenter of \( \triangle ABC \). Use the given information to find the indicated measure. (See Example 3.)

11. \( ND = 6x - 2 \)
   \( NE = 3x + 7 \)
   Find \( NF \).

12. \( NG = x + 3 \)
   \( NH = 2x - 3 \)
   Find \( NJ \).

13. \( NK = 2x - 2 \)
   \( NL = -x + 10 \)
   Find \( NM \).

14. \( NQ = 2x \)
   \( NR = 3x - 2 \)
   Find \( NS \).

15. \( P \) is the circumcenter of \( \triangle XYZ \). Use the given information to find \( PZ \).
   \( PX = 3x + 2 \)
   \( PY = 4x - 8 \)
16. \( P \) is the circumcenter of \( \triangle XYZ \). Use the given information to find \( PY \).
   \[
   PX = 4x + 3 \\
   PZ = 6x - 11
   \]

**CONSTRUCTION** In Exercises 17–20, draw a triangle of the given type. Find the circumcenter. Then construct the circumscribed circle.

17. right  
18. obtuse  
19. acute isosceles  
20. equilateral

**CONSTRUCTION** In Exercises 21–24, copy the triangle with the given angle measures. Find the incenter. Then construct the inscribed circle.

21.  
22.  
23.  
24.  

**ERROR ANALYSIS** In Exercises 25 and 26, describe and correct the error in identifying equal distances inside the triangle.

25.  
26.  

**MODELING WITH MATHEMATICS** You and two friends plan to meet to walk your dogs together. You want the meeting place to be the same distance from each person’s house. Explain how you can use the diagram to locate the meeting place. (See Example 1.)

27.  

**MODELING WITH MATHEMATICS** You are placing a fountain in a triangular koi pond. You want the fountain to be the same distance from each edge of the pond. Where should you place the fountain? Explain your reasoning. Use a sketch to support your answer. (See Example 4.)

28.  

**CRITICAL THINKING** In Exercises 29–32, complete the statement with always, sometimes, or never. Explain your reasoning.

29. The circumcenter of a scalene triangle is ______ inside the triangle.

30. If the perpendicular bisector of one side of a triangle intersects the opposite vertex, then the triangle is ______ isosceles.

31. The perpendicular bisectors of a triangle intersect at a point that is ______ equidistant from the midpoints of the sides of the triangle.

32. The angle bisectors of a triangle intersect at a point that is ______ equidistant from the sides of the triangle.
CRITICAL THINKING  In Exercises 33 and 34, find the coordinates of the circumcenter of the triangle with the given vertices.

33.  \( A(2, 5), B(6, 6), C(12, 3) \)

34.  \( D(-9, -5), E(-5, -9), F(-2, -2) \)

MATHEMATICAL CONNECTIONS  In Exercises 35 and 36, find the value of \( x \) that makes \( N \) the incenter of the triangle.

35.

\[ \begin{align*}
A & \hspace{1cm} J \\
35 & \hspace{1cm} K \\
C & \hspace{1cm} L
\end{align*} \]

36.

\[ \begin{align*}
R & \hspace{1cm} F \\
24 & \hspace{1cm} 25 \\
Q & \hspace{1cm} P
\end{align*} \]

37. PROOF  Where is the circumcenter located in any right triangle? Write a coordinate proof of this result.

38. PROVING A THEOREM  Write a proof of the Incenter Theorem.

Given  \( \triangle ABC, \overline{AD} \) bisects \( \angle CAB \), \( \overline{BD} \) bisects \( \angle CBA \), \( \overline{DE} \perp \overline{AB} \), \( \overline{DF} \perp \overline{BC} \), and \( \overline{DG} \perp \overline{CA} \)

Prove  The angle bisectors intersect at \( D \), which is equidistant from \( \overline{AB}, \overline{BC}, \) and \( \overline{CA} \).}

39. WRITING  Explain the difference between the circumcenter and the incenter of a triangle.

40. REASONING  Is the incenter of a triangle ever located outside the triangle? Explain your reasoning.

41. MODELING WITH MATHEMATICS  You are installing a circular pool in the triangular courtyard shown. You want to have the largest pool possible on the site without extending into the walkway.

![Diagram of a triangle with a pool]

a. Copy the triangle and show how to install the pool so that it just touches each edge. Then explain how you can be sure that you could not fit a larger pool on the site.

b. You want to have the largest pool possible while leaving at least 1 foot of space around the pool. Would the center of the pool be in the same position as in part (a)? Justify your answer.

42. MODELING WITH MATHEMATICS  Archaeologists find three stones. They believe that the stones were once part of a circle of stones with a community fire pit at its center. They mark the locations of stones \( A, B, \) and \( C \) on a graph, where distances are measured in feet.

![Graph with coordinates of points A, B, and C]

a. Explain how archaeologists can use a sketch to estimate the center of the circle of stones.

b. Copy the diagram and find the approximate coordinates of the point at which the archaeologists should look for the fire pit.

43. REASONING  Point \( P \) is inside \( \triangle ABC \) and is equidistant from points \( A \) and \( B \). On which of the following segments must \( P \) be located?

- \( \overline{AB} \)
- the perpendicular bisector of \( \overline{AB} \)
- \( \overline{AC} \)
- the perpendicular bisector of \( \overline{AC} \)
Chapter 6  Relationships Within Triangles

44. **CRITICAL THINKING** A high school is being built for the four towns shown on the map. Each town agrees that the school should be an equal distance from each of the four towns. Is there a single point where they could agree to build the school? If so, find it. If not, explain why not. Justify your answer with a diagram.

45. **MAKING AN ARGUMENT** Your friend says that the circumcenter of an equilateral triangle is also the incenter of the triangle. Is your friend correct? Explain your reasoning.

46. **HOW DO YOU SEE IT?** The arms of the windmill are the angle bisectors of the red triangle. What point of concurrency is the point that connects the three arms?

47. **ABSTRACT REASONING** You are asked to draw a triangle and all its perpendicular bisectors and angle bisectors.
   
   a. For which type of triangle would you need the fewest segments? What is the minimum number of segments you would need? Explain.
   
   b. For which type of triangle would you need the most segments? What is the maximum number of segments you would need? Explain.

48. **THOUGHT PROVOKING** The diagram shows an official hockey rink used by the National Hockey League. Create a triangle using hockey players as vertices in which the center circle is inscribed in the triangle. The center dot should be the incenter of your triangle. Sketch a drawing of the locations of your hockey players. Then label the actual lengths of the sides and the angle measures in your triangle.

**COMPARING METHODS** In Exercises 49 and 50, state whether you would use perpendicular bisectors or angle bisectors. Then solve the problem.

49. You need to cut the largest circle possible from an isosceles triangle made of paper whose sides are 8 inches, 12 inches, and 12 inches. Find the radius of the circle.

50. On a map of a camp, you need to create a circular walking path that connects the pool at (10, 20), the nature center at (16, 2), and the tennis court at (2, 4). Find the coordinates of the center of the circle and the radius of the circle.

51. **CRITICAL THINKING** Point D is the incenter of ΔABC. Write an expression for the length x in terms of the three side lengths AB, AC, and BC.

44. [Map of towns with coordinates]

45. [Diagram of an equilateral triangle with labels for circumcenter and incenter]

46. [Windmill diagram with labels for angle bisectors]

47. [Diagram of a triangle with perpendicular and angle bisectors]

48. [Diagram of a hockey rink with labeled points and distances]

**Maintaining Mathematical Proficiency**

The endpoints of AB are given. Find the coordinates of the midpoint M. Then find AB.

52. A(−3, 5), B(3, 5)  
53. A(2, −1), B(10, 7)  
54. A(−5, 1), B(4, −5)  
55. A(−7, 5), B(5, 9)

Write an equation of the line passing through point P that is perpendicular to the given line. Graph the equations of the lines to check that they are perpendicular.

56. P(2, 8), y = 2x + 1  
57. P(6, −3), y = −5  
58. P(−8, −6), 2x + 3y = 18  
59. P(−4, 1), y + 3 = −4(x + 3)
6.4 Medians and Altitudes of Triangles

Essential Question What conjectures can you make about the medians and altitudes of a triangle?

EXPLORATION 1 Finding Properties of the Medians of a Triangle

Work with a partner. Use dynamic geometry software. Draw any \( \triangle ABC \).

a. Plot the midpoint of \( BC \) and label it \( D \). Draw \( AD \), which is a median of \( \triangle ABC \). Construct the medians to the other two sides of \( \triangle ABC \).

b. What do you notice about the medians? Drag the vertices to change \( \triangle ABC \). Use your observations to write a conjecture about the medians of a triangle.

c. In the figure above, point \( G \) divides each median into a shorter segment and a longer segment. Find the ratio of the length of each longer segment to the length of the whole median. Is this ratio always the same? Justify your answer.

EXPLORATION 2 Finding Properties of the Altitudes of a Triangle

Work with a partner. Use dynamic geometry software. Draw any \( \triangle ABC \).

a. Construct the perpendicular segment from vertex \( A \) to \( BC \). Label the endpoint \( D \). \( AD \) is an altitude of \( \triangle ABC \).

b. Construct the altitudes to the other two sides of \( \triangle ABC \). What do you notice?

c. Write a conjecture about the altitudes of a triangle. Test your conjecture by dragging the vertices to change \( \triangle ABC \).

Communicate Your Answer

3. What conjectures can you make about the medians and altitudes of a triangle?

4. The length of median \( RU \) in \( \triangle RST \) is 3 inches. The point of concurrency of the three medians of \( \triangle RST \) divides \( RU \) into two segments. What are the lengths of these two segments?
What You Will Learn

- Use medians and find the centroids of triangles.
- Use altitudes and find the orthocenters of triangles.

Using the Median of a Triangle

A median of a triangle is a segment from a vertex to the midpoint of the opposite side. The three medians of a triangle are concurrent. The point of concurrency, called the centroid, is inside the triangle.

**Centroid Theorem**

The centroid of a triangle is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The medians of \(\triangle ABC\) meet at point \(P\), and \(AP = \frac{2}{3}AE\), \(BP = \frac{2}{3}BF\), and \(CP = \frac{2}{3}CD\).

**Proof** BigIdeasMath.com

**CONSTRUCTION** Finding the Centroid of a Triangle

Use a compass and straightedge to construct the medians of \(\triangle ABC\).

**EXAMPLE 1** Using the Centroid of a Triangle

In \(\triangle RST\), point \(Q\) is the centroid, and \(SQ = 8\). Find \(QW\) and \(SW\).

**SOLUTION**

\[ SQ = \frac{2}{3}SW \quad \text{Centroid Theorem} \]
\[ 8 = \frac{2}{3}SW \quad \text{Substitute 8 for } SQ. \]
\[ 12 = SW \quad \text{Multiply each side by the reciprocal, } \frac{3}{2}. \]

Then \(QW = SW - SQ = 12 - 8 = 4\).

- So, \(QW = 4\) and \(SW = 12\).
Finding the Centroid of a Triangle

Find the coordinates of the centroid of \( \triangle RST \) with vertices \( R(2, 1) \), \( S(5, 8) \), and \( T(8, 3) \).

**SOLUTION**

**Step 1** Graph \( \triangle RST \).

**Step 2** Use the Midpoint Formula to find the midpoint \( V \) of \( RT \) and sketch median \( SV \).

\[
V\left(\frac{2 + 8}{2}, \frac{1 + 3}{2}\right) = (5, 2)
\]

**Step 3** Find the centroid. It is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The distance from vertex \( S(5, 8) \) to \( V(5, 2) \) is \( 8 - 2 = 6 \) units. 
So, the centroid is \( \frac{2}{3}(6) = 4 \) units down from vertex \( S \) on \( SV \).

So, the coordinates of the centroid \( P \) are \( (5, 8 - 4) \), or \( (5, 4) \).

**Monitoring Progress**

There are three paths through a triangular park. Each path goes from the midpoint of one edge to the opposite corner. The paths meet at point \( P \).

1. Find \( PS \) and \( PC \) when \( SC = 2100 \) feet.
2. Find \( TC \) and \( BC \) when \( BT = 1000 \) feet.
3. Find \( PA \) and \( TA \) when \( PT = 800 \) feet.

Find the coordinates of the centroid of the triangle with the given vertices.

4. \( F(2, 5), G(4, 9), H(6, 1) \)
5. \( X(-3, 3), Y(1, 5), Z(-1, -2) \)

**Using the Altitude of a Triangle**

An altitude of a triangle is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side.

**Core Concept**

**Orthocenter**

The lines containing the altitudes of a triangle are concurrent. This point of concurrency is the orthocenter of the triangle.

The lines containing \( AF, BD, \) and \( CE \) meet at the orthocenter \( G \) of \( \triangle ABC \).
As shown below, the location of the orthocenter $P$ of a triangle depends on the type of triangle.

- Acute triangle: $P$ is inside triangle.
- Right triangle: $P$ is on triangle.
- Obtuse triangle: $P$ is outside triangle.

**Example 3** Finding the Orthocenter of a Triangle

Find the coordinates of the orthocenter of $\triangle XYZ$ with vertices $X(-5, -1)$, $Y(-2, 4)$, and $Z(3, -1)$.

**Solution**

**Step 1** Graph $\triangle XYZ$.

**Step 2** Find an equation of the line that contains the altitude from $Y$ to $\overline{XZ}$. Because $\overline{XZ}$ is horizontal, the altitude is vertical. The line that contains the altitude passes through $Y(-2, 4)$. So, the equation of the line is $x = -2$.

**Step 3** Find an equation of the line that contains the altitude from $X$ to $\overline{YZ}$. The slope of $\overline{YZ}$ is $\frac{-1 - 4}{3 - (-2)} = -1$.

Because the product of the slopes of two perpendicular lines is $-1$, the slope of a line perpendicular to $\overline{YZ}$ is 1. The line passes through $X(-5, -1)$.

\[ y = mx + b \]

\[ -1 = 1(-5) + b \] Substitute $-1$ for $y$, 1 for $m$, and $-5$ for $x$.

\[ 4 = b \] Solve for $b$.

So, the equation of the line is $y = x + 4$.

**Step 4** Find the point of intersection of the graphs of the equations $x = -2$ and $y = x + 4$.

Substitute $-2$ for $x$ in the equation $y = x + 4$. Then solve for $y$.

\[ y = x + 4 \] Write equation.

\[ y = -2 + 4 \] Substitute $-2$ for $x$.

\[ y = 2 \] Solve for $y$.

So, the coordinates of the orthocenter are $(-2, 2)$.

**Monitoring Progress**

Tell whether the orthocenter of the triangle with the given vertices is inside, on, or outside the triangle. Then find the coordinates of the orthocenter.

6. $A(0, 3), B(0, -2), C(6, -3)$
7. $J(-3, -4), K(-3, 4), L(5, 4)$
In an isosceles triangle, the perpendicular bisector, angle bisector, median, and altitude from the vertex angle to the base are all the same segment. In an equilateral triangle, this is true for any vertex.

**EXAMPLE 4** Proving a Property of Isosceles Triangles

Prove that the median from the vertex angle to the base of an isosceles triangle is an altitude.

**SOLUTION**

Given $\triangle ABC$ is isosceles, with base $\overline{AC}$.
$BD$ is the median to base $\overline{AC}$.

Prove $BD$ is an altitude of $\triangle ABC$.

**Paragraph Proof** Legs $\overline{AB}$ and $\overline{BC}$ of isosceles $\triangle ABC$ are congruent. $\overline{CD} \cong \overline{AD}$ because $BD$ is the median to $\overline{AC}$. Also, $\overline{BD} \cong \overline{BD}$ by the Reflexive Property of Congruence. So, $\triangle ABD \cong \triangle CBD$ by the SSS Congruence Theorem. $\angle ADB \cong \angle CDB$ because corresponding parts of congruent triangles are congruent. Also, $\angle ADB$ and $\angle CDB$ are a linear pair. $BD$ and $AC$ intersect to form a linear pair of congruent angles, so $BD \perp AC$ and $BD$ is an altitude of $\triangle ABC$.

**Monitoring Progress**

8. **WHAT IF?** In Example 4, you want to show that median $BD$ is also an angle bisector. How would your proof be different?

**Concept Summary**

<table>
<thead>
<tr>
<th>Segments, Lines, Rays, and Points in Triangles</th>
<th>Example</th>
<th>Point of Concurrency</th>
<th>Property</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>perpendicular bisector</td>
<td><img src="image1.png" alt="Perpendicular Bisector" /></td>
<td>circumcenter</td>
<td>The circumcenter $P$ of a triangle is equidistant from the vertices of the triangle.</td>
<td><img src="image2.png" alt="Circumcenter" /></td>
</tr>
<tr>
<td>angle bisector</td>
<td><img src="image3.png" alt="Angle Bisector" /></td>
<td>incenter</td>
<td>The incenter $I$ of a triangle is equidistant from the sides of the triangle.</td>
<td><img src="image4.png" alt="Incenter" /></td>
</tr>
<tr>
<td>median</td>
<td><img src="image5.png" alt="Median" /></td>
<td>centroid</td>
<td>The centroid $R$ of a triangle is two thirds of the distance from each vertex to the midpoint of the opposite side.</td>
<td><img src="image6.png" alt="Centroid" /></td>
</tr>
<tr>
<td>altitude</td>
<td><img src="image7.png" alt="Altitude" /></td>
<td>orthocenter</td>
<td>The lines containing the altitudes of a triangle are concurrent at the orthocenter $O$.</td>
<td><img src="image8.png" alt="Orthocenter" /></td>
</tr>
</tbody>
</table>
6.4 Exercises

Vocabulary and Core Concept Check

1. **VOCABULARY** Name the four types of points of concurrency. Which lines intersect to form each of the points?

2. **COMPLETE THE SENTENCE** The length of a segment from a vertex to the centroid is __________ the length of the median from that vertex.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, point $P$ is the centroid of $\triangle LMN$. Find $PN$ and $QP$. (See Example 1.)

3. $QN = 9$

![Triangle LMN with point P as the centroid.]

4. $QN = 21$

5. $QN = 30$

6. $QN = 42$

In Exercises 7–10, point $D$ is the centroid of $\triangle ABC$. Find $CD$ and $CE$.

7. $DE = 5$

![Triangle ABC with point D as the centroid.]

8. $DE = 11$

9. $DE = 9$

10. $DE = 15$

In Exercises 11–14, point $G$ is the centroid of $\triangle ABC$. $BG = 6$, $AF = 12$, and $AE = 15$. Find the length of the segment.

11. $\overline{FC}$

![Triangle ABC with point G as the centroid.]

12. $\overline{BF}$

13. $\overline{AG}$

14. $\overline{GE}$

In Exercises 15–18, find the coordinates of the centroid of the triangle with the given vertices. (See Example 2.)

15. $A(2, 3), B(8, 1), C(5, 7)$

16. $F(1, 5), G(–2, 7), H(–6, 3)$

17. $S(5, 5), T(11, −3), U(–1, 1)$

18. $X(1, 4), Y(7, 2), Z(2, 3)$

In Exercises 19–22, tell whether the orthocenter is inside, on, or outside the triangle. Then find the coordinates of the orthocenter. (See Example 3.)

19. $L(0, 5), M(3, 1), N(8, 1)$

20. $X(–3, 2), Y(5, 2), Z(–3, 6)$

21. $A(–4, 0), B(1, 0), C(–1, 3)$

22. $T(–2, 1), U(2, 1), V(0, 4)$

CONSTRUCTION In Exercises 23–26, draw the indicated triangle and find its centroid and orthocenter.

23. isosceles right triangle

24. obtuse scalene triangle

25. right scalene triangle

26. acute isosceles triangle
ERROR ANALYSIS  In Exercises 27 and 28, describe and correct the error in finding $DE$. Point $D$ is the centroid of $\triangle ABC$.

27. $DE = \frac{2}{3}AE$  \(\times\)
   $DE = \frac{2}{3}(18)$
   $DE = 12$

28. $DE = \frac{2}{3}AD$  \(\times\)
   $AD = 24$
   $DE = \frac{2}{3}(24)$
   $DE = 16$  \(\times\)

PROOF  In Exercises 29 and 30, write a proof of the statement. (See Example 4.)

29. The angle bisector from the vertex angle to the base of an isosceles triangle is also a median.

30. The altitude from the vertex angle to the base of an isosceles triangle is also a perpendicular bisector.

CRITICAL THINKING  In Exercises 31–36, complete the statement with always, sometimes, or never. Explain your reasoning.

31. The centroid is _________ on the triangle.

32. The orthocenter is _________ outside the triangle.

33. A median is _________ the same line segment as a perpendicular bisector.

34. An altitude is _________ the same line segment as an angle bisector.

35. The centroid and orthocenter are _________ the same point.

36. The centroid is _________ formed by the intersection of the three medians.

37. WRITING  Compare an altitude of a triangle with a perpendicular bisector of a triangle.

38. WRITING  Compare a median, an altitude, and an angle bisector of a triangle.

39. MODELING WITH MATHEMATICS  Find the area of the triangular part of the paper airplane wing that is outlined in red. Which special segment of the triangle did you use?

40. ANALYZING RELATIONSHIPS  Copy and complete the statement for $\triangle DEF$ with centroid $K$ and medians $DH, EJ,$ and $FG$.
   a. $EJ = ______ KJ$  \(\times\)
   b. $DK = ______ KH$
   c. $FG = ______ KF$  \(\times\)
   d. $KG = ______ FG$

MATHEMATICAL CONNECTIONS  In Exercises 41–44, point $D$ is the centroid of $\triangle ABC$. Use the given information to find the value of $x$.

41. $BD = 4x + 5$ and $BF = 9x$

42. $GD = 2x - 8$ and $GC = 3x + 3$

43. $AD = 5x$ and $DE = 3x - 2$

44. $DF = 4x - 1$ and $BD = 6x + 4$

45. MATHEMATICAL CONNECTIONS  Graph the lines on the same coordinate plane. Find the centroid of the triangle formed by their intersections.
   \[y_1 = 3x - 4\]
   \[y_2 = \frac{3}{4}x + 5\]
   \[y_3 = -\frac{3}{2}x - 4\]

46. CRITICAL THINKING  In what type(s) of triangles can a vertex be one of the points of concurrency of the triangle? Explain your reasoning.
47. **Writing Equations** Use the numbers and symbols to write three different equations for $PE$.

![Diagram of a triangle with points A, B, C, D, E, F, and P]

$PE = \frac{1}{4} + \frac{1}{3} - \frac{1}{2} + \frac{2}{3}$

48. **How Do You See It?** Use the figure.

![Diagram of a triangle with points J, K, L, M, and N]

- a. What type of segment is $KM$? Which point of concurrency lies on $KM$?
- b. What type of segment is $KN$? Which point of concurrency lies on $KN$?
- c. Compare the areas of $\triangle JKM$ and $\triangle KLM$. Do you think the areas of the triangles formed by the median of any triangle will always compare this way? Explain your reasoning.

49. **Making an Argument** Your friend claims that it is possible for the circumcenter, incenter, centroid, and orthocenter to all be the same point. Do you agree? Explain your reasoning.

50. **Drawing Conclusions** The center of gravity of a triangle, the point where a triangle can balance on the tip of a pencil, is one of the four points of concurrency. Draw and cut out a large scalene triangle on a piece of cardboard. Which of the four points of concurrency is the center of gravity? Explain.

51. **Proof** Prove that a median of an equilateral triangle is also an angle bisector, perpendicular bisector, and altitude.

52. **Thought Provoking** Construct an acute scalene triangle. Find the orthocenter, centroid, and circumcenter. What can you conclude about the three points of concurrency?

53. **Construction** Follow the steps to construct a nine-point circle. Why is it called a nine-point circle?

**Step 1** Construct a large acute scalene triangle.

**Step 2** Find the orthocenter and circumcenter of the triangle.

**Step 3** Find the midpoint between the orthocenter and circumcenter.

**Step 4** Find the midpoint between each vertex and the orthocenter.

**Step 5** Construct a circle. Use the midpoint in Step 3 as the center of the circle, and the distance from the center to the midpoint of a side of the triangle as the radius.

54. **Proof** Prove the statements in parts (a)–(c).

**Given** $LP$ and $MQ$ are medians of scalene $\triangle LMN$.

Point $R$ is on $LP$ such that $LP \cong PR$. Point $S$ is on $MQ$ such that $MQ \cong QS$.

**Prove**

- a. $\overline{NS} \cong \overline{NR}$
- b. $\overline{NS}$ and $\overline{NR}$ are both parallel to $LM$.
- c. $R$, $N$, and $S$ are collinear.

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Determine whether $\overline{AB}$ is parallel to $\overline{CD}$. **(Skills Review Handbook)**

55. $A(5, 6), B(−1, 3), C(−4, 9), D(−16, 3)$

56. $A(−3, 6), B(5, 4), C(−14, −10), D(−2, −7)$

57. $A(6, −3), B(5, 2), C(−4, −4), D(−5, 2)$

58. $A(−5, 6), B(−7, 2), C(7, 1), D(4, −5)$

---

Chapter 6  Relationships Within Triangles

Copyright © Big Ideas Learning, LLC. All rights reserved.
6.1–6.4 What Did You Learn?

Core Vocabulary

- proof, p. 336
- two-column proof, p. 336
- paragraph proof, p. 337
- flowchart proof, or flow proof, p. 338
- coordinate proof, p. 339
- equidistant, p. 344
- concurrent, p. 352
- point of concurrency, p. 352
- circumcenter, p. 352
- incenter, p. 355
- median of a triangle, p. 362
- centroid, p. 362
- altitude of a triangle, p. 363
- orthocenter, p. 363

Core Concepts

Section 6.1
- Writing Two-Column Proofs, p. 336
- Writing Paragraph Proofs, p. 337
- Writing Flowchart Proofs, p. 338
- Writing Coordinate Proofs, p. 339

Section 6.2
- Perpendicular Bisector Theorem, p. 344
- Converse of the Perpendicular Bisector Theorem, p. 344
- Angle Bisector Theorem, p. 346
- Converse of the Angle Bisector Theorem, p. 346

Section 6.3
- Circumcenter Theorem, p. 352
- Incenter Theorem, p. 355

Section 6.4
- Centroid Theorem, p. 362
- Orthocenter, p. 363
- Segments, Lines, Rays, and Points in Triangles, p. 365

Mathematical Practices

1. Did you make a plan before completing your proof in Exercise 37 on page 350? Describe your thought process.

2. What tools did you use to complete Exercises 17–20 on page 358? Describe how you could use technological tools to complete these exercises.

Rework Your Notes

A good way to reinforce concepts and put them into your long-term memory is to rework your notes. When you take notes, leave extra space on the pages. You can go back after class and fill in

- important definitions and rules,
- additional examples, and
- questions you have about the material.
6.1–6.4 Quiz

Find the indicated measure. Explain your reasoning. *(Section 6.2)*

1. \( UV \)

2. \( QP \)

3. \( m\angle GJK \)

Find the coordinates of the circumcenter of the triangle with the given vertices. *(Section 6.3)*

4. \( A(-4, 2), B(-4, -4), C(0, -4) \)

5. \( D(3, 5), E(7, 9), F(11, 5) \)

The incenter of \( \triangle ABC \) is point \( N \). Use the given information to find the indicated measure. *(Section 6.3)*

6. \( NQ = 2x + 1, NR = 4x - 9 \)
   Find \( NS \).

7. \( NU = -3x + 6, NV = -5x \)
   Find \( NT \).

8. \( NZ = 4x - 10, NY = 3x - 1 \)
   Find \( NW \).

Find the coordinates of the centroid of the triangle with the given vertices. *(Section 6.4)*

9. \( J(-1, 2), K(5, 6), L(5, -2) \)

10. \( M(-8, -6), N(-4, -2), P(0, -4) \)

Tell whether the orthocenter is inside, on, or outside the triangle. Then find its coordinates. *(Section 6.4)*

11. \( T(-2, 5), U(0, 1), V(2, 5) \)

12. \( X(-1, -4), Y(7, -4), Z(7, 4) \)

13. A woodworker is cutting the largest wheel possible from a triangular scrap of wood. The wheel just touches each side of the triangle, as shown. *(Section 6.1 and Section 6.3)*

   a. Which point of concurrency is the center of the circle? What type of segments are \( BG, CG, \) and \( AG \)?
   
   b. Prove that \( \triangle BGF \cong \triangle BGE \).
   
   c. Find the radius of the wheel to the nearest tenth of a centimeter. Justify your answer.

14. The Deer County Parks Committee plans to build a park at point \( P \), equidistant from the three largest cities labeled \( X, Y, \) and \( Z \). The map shown was created by the committee. *(Section 6.3 and Section 6.4)*

   a. Which point of concurrency did the committee use as the location of the park?
   
   b. Did the committee use the best point of concurrency for the location of the park? If not, which point would be better to use? Explain.
6.5 The Triangle Midsegment Theorem

**Essential Question** How are the midsegments of a triangle related to the sides of the triangle?

**EXPLORATION 1 Midsegments of a Triangle**

Work with a partner. Use dynamic geometry software. Draw any \( \triangle ABC \).

a. Plot midpoint \( D \) of \( AB \) and midpoint \( E \) of \( BC \). Draw \( DE \), which is a midsegment of \( \triangle ABC \).

b. Compare the slope and length of \( DE \) with the slope and length of \( AC \).

c. Write a conjecture about the relationships between the midsegments and sides of a triangle. Test your conjecture by drawing the other midsegments of \( \triangle ABC \), dragging vertices to change \( \triangle ABC \), and noting whether the relationships hold.

**CONSTRUCTING VIABLE ARGUMENTS**

To be proficient in math, you need to make conjectures and build a logical progression of statements to explore the truth of your conjectures.

**EXPLORATION 2 Midsegments of a Triangle**

Work with a partner. Use dynamic geometry software. Draw any \( \triangle ABC \).

a. Draw all three midsegments of \( \triangle ABC \).

b. Use the drawing to write a conjecture about the triangle formed by the midsegments of the original triangle.

**Communicate Your Answer**

3. How are the midsegments of a triangle related to the sides of the triangle?

4. In \( \triangle RST, \overline{UV} \) is the midsegment connecting the midpoints of \( \overline{RS} \) and \( \overline{ST} \). Given \( UV = 12 \), find \( RT \).
What You Will Learn

- Use midsegments of triangles in the coordinate plane.
- Use the Triangle Midsegment Theorem to find distances.

Using the Midsegment of a Triangle

A **midsegment of a triangle** is a segment that connects the midpoints of two sides of the triangle. Every triangle has three midsegments, which form the **midsegment triangle**.

The midsegments of \( \triangle ABC \) at the right are \( \overline{MP} \), \( \overline{MN} \), and \( \overline{NP} \). The **midsegment triangle** is \( \triangle MNP \).

**EXAMPLE 1** Using Midsegments in the Coordinate Plane

In \( \triangle JKL \), show that midsegment \( \overline{MN} \) is parallel to \( \overline{JL} \) and that \( MN = \frac{1}{2} JL \).

**SOLUTION**

**Step 1** Find the coordinates of \( M \) and \( N \) by finding the midpoints of \( \overline{JK} \) and \( \overline{KL} \).

\[
M \left( \frac{-6 + (2)}{2}, \frac{1 + 5}{2} \right) = M \left( -4, 3 \right) \]

\[
N \left( \frac{-2 + 2}{2}, \frac{5 + (-1)}{2} \right) = N \left( 0, 2 \right) \]

**Step 2** Find and compare the slopes of \( \overline{MN} \) and \( \overline{JL} \).

slope of \( \overline{MN} = \frac{2 - 3}{0 - (-4)} = -\frac{1}{4} \)

slope of \( \overline{JL} = \frac{-1 - 1}{2 - (-6)} = -\frac{2}{8} = -\frac{1}{4} \)

- Because the slopes are the same, \( \overline{MN} \) is parallel to \( \overline{JL} \).

**Step 3** Find and compare the lengths of \( \overline{MN} \) and \( \overline{JL} \).

\[
MN = \sqrt{[0 - (-4)]^2 + (2 - 3)^2} = \sqrt{16 + 1} = \sqrt{17} \]

\[
JL = \sqrt{[2 - (-6)]^2 + (-1 - 1)^2} = \sqrt{64 + 4} = \sqrt{68} = 2\sqrt{17} \]

- Because \( \sqrt{17} = \frac{1}{2} (2\sqrt{17}) \), \( MN = \frac{1}{2} JL \).

**Monitoring Progress**

Use the graph of \( \triangle ABC \).

1. In \( \triangle ABC \), show that midsegment \( \overline{DE} \) is parallel to \( \overline{AC} \) and that \( DE = \frac{1}{2} AC \).

2. Find the coordinates of the endpoints of midsegment \( \overline{EF} \), which is opposite \( \overline{AB} \). Show that \( EF \) is parallel to \( \overline{AB} \) and that \( EF = \frac{1}{2} AB \).
Using the Triangle Midsegment Theorem

**Triangle Midsegment Theorem**

The segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half as long as that side.

\[ DE \] is a midsegment of \( \triangle ABC \), \( DE \parallel AC \), and \( DE = \frac{1}{2} AC \).

**Proof** Example 2, p. 373; Monitoring Progress Question 3, p. 373; Ex. 22, p. 376

---

**EXAMPLE 2** Proving the Triangle Midsegment Theorem

Write a coordinate proof of the Triangle Midsegment Theorem for one midsegment.

**Given** \( \overline{DE} \) is a midsegment of \( \triangle OBC \).

**Prove** \( \overline{DE} \parallel \overline{OC} \) and \( DE = \frac{1}{2} OC \)

**SOLUTION**

**Step 1** Place \( \triangle OBC \) in a coordinate plane and assign coordinates. Because you are finding midpoints, use \( 2p, 2q, \) and \( 2r \). Then find the coordinates of \( D \) and \( E \).

\[
D \left( \frac{2q + 0}{2}, \frac{2r + 0}{2} \right) = D(q, r) \quad E \left( \frac{2q + 2p}{2}, \frac{2r + 0}{2} \right) = E(q + p, r)
\]

**Step 2** Prove \( \overline{DE} \parallel \overline{OC} \). The \( y \)-coordinates of \( D \) and \( E \) are the same, so \( \overline{DE} \) has a slope of 0. \( \overline{OC} \) is on the \( x \)-axis, so its slope is 0.

\[ \text{Because their slopes are the same, } \overline{DE} \parallel \overline{OC}. \]

**Step 3** Prove \( DE = \frac{1}{2} OC \). Use the Ruler Postulate to find \( DE \) and \( OC \).

\[ DE = \left| (q + p) - q \right| = p \]
\[ OC = \left| 2p - 0 \right| = 2p \]

\[ \text{Because } p = \frac{1}{2}(2p), \text{ } DE = \frac{1}{2} OC. \]

---

**Monitoring Progress**

3. In Example 2, find the coordinates of \( F \), the midpoint of \( \overline{OC} \). Show that \( \overline{FE} \parallel \overline{OB} \)

and \( FE = \frac{1}{2} OB \).

---

**EXAMPLE 3** Using the Triangle Midsegment Theorem

Triangles are used for strength in roof trusses. In the diagram, \( \overline{UV} \) and \( \overline{VW} \) are midsegments of \( \triangle RST \). Find \( \overline{UV} \) and \( \overline{RS} \).

**SOLUTION**

\[
UV = \frac{1}{2} \cdot RT = \frac{1}{2} \cdot (90 \text{ in.}) = 45 \text{ in.}
\]
\[
RS = 2 \cdot VW = 2(57 \text{ in.}) = 114 \text{ in.}
\]
**EXAMPLE 4**  
**Using the Triangle Midsegment Theorem**

In the kaleidoscope image, $\overline{AE} \cong \overline{BE}$ and $\overline{AD} \cong \overline{CD}$.

Show that $\overline{CB} || \overline{DE}$.

**SOLUTION**

Because $\overline{AE} \cong \overline{BE}$ and $\overline{AD} \cong \overline{CD}$, $E$ is the midpoint of $\overline{AB}$ and $D$ is the midpoint of $\overline{AC}$ by definition. Then $\overline{DE}$ is a midsegment of $\triangle ABC$ by definition and $\overline{CB} || \overline{DE}$ by the Triangle Midsegment Theorem.

**EXAMPLE 5**  
**Modeling with Mathematics**

Pear Street intersects Cherry Street and Peach Street at their midpoints. Your home is at point $P$. You leave your home and jog down Cherry Street to Plum Street, over Plum Street to Peach Street, up Peach Street to Pear Street, over Pear Street to Cherry Street, and then back home up Cherry Street. About how many miles do you jog?

**SOLUTION**

1. **Understand the Problem**  
   You know the distances from your home to Plum Street along Peach Street, from Peach Street to Cherry Street along Plum Street, and fromPear Street to your home along Cherry Street. You need to find the other distances on your route, then find the total number of miles you jog.

2. **Make a Plan**  
   By definition, you know that Pear Street is a midsegment of the triangle formed by the other three streets. Use the Triangle Midsegment Theorem to find the length of Pear Street and the definition of midsegment to find the length of Cherry Street. Then add the distances along your route.

3. **Solve the Problem**

   length of Pear Street $= \frac{1}{2} \cdot$ (length of Plum St.) $= \frac{1}{2} (1.4 \text{ mi}) = 0.7 \text{ mi}$

   length of Cherry Street $= 2 \cdot$ (length from $P$ to Pear St.) $= 2(1.3 \text{ mi}) = 2.6 \text{ mi}$

   distance along your route: $2.6 + 1.4 + \frac{1}{2}(2.25) + 0.7 + 1.3 = 7.125$

   So, you jog about 7 miles.

4. **Look Back**  
   Use compatible numbers to check that your answer is reasonable.

   total distance:
   
   $2.6 + 1.4 + \frac{1}{2}(2.25) + 0.7 + 1.3 \approx 2.5 + 1.5 + 0.5 + 1.5 = 7 \, \checkmark$

**Monitoring Progress**

**4.** Copy the diagram in Example 3. Draw and name the third midsegment. Then find the length of $\overline{VS}$ when the length of the third midsegment is 81 inches.

**5.** In Example 4, if $F$ is the midpoint of $\overline{CB}$, what do you know about $\overline{DF}$?

**6.** WHAT IF? In Example 5, you jog down Peach Street to Plum Street, over Plum Street to Cherry Street, up Cherry Street to Pear Street, over Pear Street to Peach Street, and then back home up Peach Street. Do you jog more miles in Example 5? Explain.
Vocabulary and Core Concept Check

1. **VOCABULARY** The __________ of a triangle is a segment that connects the midpoints of two sides of the triangle.

2. **COMPLETE THE SENTENCE** If $\overline{DE}$ is the midsegment opposite $\overline{AC}$ in $\triangle ABC$, then $\overline{DE} \parallel \overline{AC}$ and $DE = \frac{1}{2}AC$ by the Triangle Midsegment Theorem.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, use the graph of $\triangle ABC$ with midsegments $\overline{DE}, \overline{EF}$, and $\overline{DF}$. (See Example 1.)

3. Find the coordinates of points $D, E,$ and $F$.

4. Show that $\overline{DE}$ is parallel to $\overline{CB}$ and that $DE = \frac{1}{2}CB$.

5. Show that $\overline{EF}$ is parallel to $\overline{AC}$ and that $EF = \frac{1}{2}AC$.

6. Show that $\overline{DF}$ is parallel to $\overline{AB}$ and that $DF = \frac{1}{2}AB$.

In Exercises 7–10, $\overline{DE}$ is a midsegment of $\triangle ABC$. Find the value of $x$. (See Example 3.)

7. 

8. 

9. 

10. 

In Exercises 11–16, $\overline{XJ} \cong \overline{JY}, \overline{YL} \cong \overline{LZ},$ and $\overline{XK} \cong \overline{KZ}$. Copy and complete the statement. (See Example 4.)

11. $\overline{JK} \parallel ____$

12. $\overline{JL} \parallel ____$

13. $\overline{XY} \parallel ____$

14. $\overline{JY} \cong ____ \cong ____$

15. $\overline{JL} \cong ____ \cong ____$

16. $\overline{JK} \cong ____ \cong ____$

MATHEMATICAL CONNECTIONS In Exercises 17–19, use $\triangle GHJ$, where $A, B,$ and $C$ are midpoints of the sides.

17. When $AB = 3x + 8$ and $GJ = 2x + 24$, what is $AB$?

18. When $AC = 3y - 5$ and $HJ = 4y + 2$, what is $HB$?

19. When $GH = 7z - 1$ and $CB = 4z - 3$, what is $GA$?

20. ERROR ANALYSIS Describe and correct the error.

$DE = \frac{1}{2}BC,$ so by the Triangle Midsegment Theorem, $\overline{AD} \cong \overline{DB}$ and $\overline{AE} \cong \overline{EC}$. 

---

Copyright © Big Ideas Learning, LLC. All rights reserved.
21. **MODELING WITH MATHEMATICS** The distance between consecutive bases on a baseball field is 90 feet. A second baseman stands halfway between first base and second base, a shortstop stands halfway between second base and third base, and a pitcher stands halfway between first base and third base. Find the distance between the shortstop and the pitcher. *(See Example 5.)*

22. **PROVING A THEOREM** Use the figure from Example 2 to prove the Triangle Midsegment Theorem for midsegment $DF$, where $F$ is the midpoint of $OC$. *(See Example 2.)*

23. **CRITICAL THINKING** $XY$ is a midsegment of $\triangle LMN$. Suppose $DE$ is called a “quarter segment” of $\triangle LMN$. What do you think an “eighth segment” would be? Make conjectures about the properties of a quarter segment and an eighth segment. Use variable coordinates to verify your conjectures.

24. **THOUGHT PROVOKING** Find a real-life object that uses midsegments as part of its structure. Print a photograph of the object and identify the midsegments of one of the triangles in the structure.

25. **ABSTRACT REASONING** To create the design shown, shade the triangle formed by the three midsegments of the triangle. Then repeat the process for each unshaded triangle.

![Stage 0 and Stage 1](image)

![Stage 2 and Stage 3](image)

- **a.** What is the perimeter of the shaded triangle in Stage 1?
- **b.** What is the total perimeter of all the shaded triangles in Stage 2?
- **c.** What is the total perimeter of all the shaded triangles in Stage 3?

26. **HOW DO YOU SEE IT?** Explain how you know that the yellow triangle is the midsegment triangle of the red triangle in the pattern of floor tiles shown.

27. **ATTENDING TO PRECISION** The points $P(2, 1)$, $Q(4, 5)$, and $R(7, 4)$ are the midpoints of the sides of a triangle. Graph the three midsegments. Then show how to use your graph and the properties of midsegments to draw the original triangle. Give the coordinates of each vertex.

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Find a counterexample to show that the conjecture is false. *(Skills Review Handbook)*

28. The difference of two numbers is always less than the greater number.
29. An isosceles triangle is always equilateral.
**Essential Question** How are the sides related to the angles of a triangle? How are any two sides of a triangle related to the third side?

**EXPLORATION 1** Comparing Angle Measures and Side Lengths

Work with a partner. Use dynamic geometry software. Draw any scalene \( \triangle ABC \).

a. Find the side lengths and angle measures of the triangle.

Sample

- **Points**
  - \( A(1, 3) \)
  - \( B(5, 1) \)
  - \( C(7, 4) \)

- **Angles**
  - \( m \angle A = ? \)
  - \( m \angle B = ? \)
  - \( m \angle C = ? \)

- **Segments**
  - \( BC = ? \)
  - \( AC = ? \)
  - \( AB = ? \)

b. Order the side lengths. Order the angle measures. What do you observe?

c. Drag the vertices of \( \triangle ABC \) to form new triangles. Record the side lengths and angle measures in a table. Write a conjecture about your findings.

**EXPLORATION 2** A Relationship of the Side Lengths of a Triangle

Work with a partner. Use dynamic geometry software. Draw any \( \triangle ABC \).

a. Find the side lengths of the triangle.

b. Compare each side length with the sum of the other two side lengths.

Sample

- **Points**
  - \( A(0, 2) \)
  - \( B(2, -1) \)
  - \( C(5, 3) \)

- **Segments**
  - \( BC = ? \)
  - \( AC = ? \)
  - \( AB = ? \)

c. Drag the vertices of \( \triangle ABC \) to form new triangles and repeat parts (a) and (b). Organize your results in a table. Write a conjecture about your findings.

**Communicate Your Answer**

3. How are the sides related to the angles of a triangle? How are any two sides of a triangle related to the third side?

4. Is it possible for a triangle to have side lengths of 3, 4, and 10? Explain.
What You Will Learn

- Write indirect proofs.
- List sides and angles of a triangle in order by size.
- Use the Triangle Inequality Theorem to find possible side lengths of triangles.

Writing an Indirect Proof

Suppose a student looks around the cafeteria, concludes that hamburgers are not being served, and explains as follows.

At first, I assumed that we are having hamburgers because today is Tuesday, and Tuesday is usually hamburger day.

There is always ketchup on the table when we have hamburgers, so I looked for the ketchup, but I didn’t see any.

So, my assumption that we are having hamburgers must be false.

The student uses indirect reasoning. In an indirect proof, you start by making the temporary assumption that the desired conclusion is false. By then showing that this assumption leads to a logical impossibility, you prove the original statement true by contradiction.

How to Write an Indirect Proof (Proof by Contradiction)

Step 1 Identify the statement you want to prove. Assume temporarily that this statement is false by assuming that its opposite is true.

Step 2 Reason logically until you reach a contradiction.

Step 3 Point out that the desired conclusion must be true because the contradiction proves the temporary assumption false.

EXAMPLE 1 Writing an Indirect Proof

Write an indirect proof that in a given triangle, there can be at most one right angle.

Given \( \triangle ABC \)

Prove \( \triangle ABC \) can have at most one right angle.

SOLUTION

Step 1 Assume temporarily that \( \triangle ABC \) has two right angles. Then assume \( \angle A \) and \( \angle B \) are right angles.

Step 2 By the definition of right angle, \( m\angle A = m\angle B = 90^\circ \). By the Triangle Sum Theorem, \( m\angle A + m\angle B + m\angle C = 180^\circ \). Using the Substitution Property of Equality, \( 90^\circ + 90^\circ + m\angle C = 180^\circ \). So, \( m\angle C = 0^\circ \) by the Subtraction Property of Equality. A triangle cannot have an angle measure of \( 0^\circ \). So, this contradicts the given information.

Step 3 So, the assumption that \( \triangle ABC \) has two right angles must be false, which proves that \( \triangle ABC \) can have at most one right angle.

Monitoring Progress

Help in English and Spanish at BigIdeasMath.com

1. Write an indirect proof that a scalene triangle cannot have two congruent angles.
Relating Sides and Angles of a Triangle

**EXAMPLE 2** Relating Side Length and Angle Measure

Draw an obtuse scalene triangle. Find the largest angle and longest side and mark them in red. Find the smallest angle and shortest side and mark them in blue. What do you notice?

**SOLUTION**

The longest side and largest angle are opposite each other.

The shortest side and smallest angle are opposite each other.

The relationships in Example 2 are true for all triangles, as stated in the two theorems below. These relationships can help you decide whether a particular arrangement of side lengths and angle measures in a triangle may be possible.

**Theorems**

**Triangle Longer Side Theorem**

If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

*Proof* Ex. 43, p. 384

**Triangle Larger Angle Theorem**

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.

*Proof* p. 379

**COMMON ERROR**

Be careful not to confuse the symbol \( \angle \) meaning angle with the symbol \( < \) meaning is less than. Notice that the bottom edge of the angle symbol is horizontal.

**COMMON ERROR**

Be sure to consider all cases when assuming the opposite is true.

**PROOF** Triangle Larger Angle Theorem

**Given** \( m\angle A > m\angle C \)

**Prove** \( BC > AB \)

**Indirect Proof**

**Step 1** Assume temporarily that \( BC < AB \). Then it follows that either \( BC < AB \) or \( BC = AB \).

**Step 2** If \( BC < AB \), then \( m\angle A < m\angle C \) by the Triangle Longer Side Theorem. If \( BC = AB \), then \( m\angle A = m\angle C \) by the Base Angles Theorem.

**Step 3** Both conclusions contradict the given statement that \( m\angle A > m\angle C \). So, the temporary assumption that \( BC < AB \) cannot be true. This proves that \( BC > AB \).
EXAMPLE 3  Ordering Angle Measures of a Triangle

You are constructing a stage prop that shows a large triangular mountain. The bottom edge of the mountain is about 32 feet long, the left slope is about 24 feet long, and the right slope is about 26 feet long. List the angles of \( \triangle JKL \) in order from smallest to largest.

**SOLUTION**

Draw the triangle that represents the mountain. Label the side lengths.

The sides from shortest to longest are \( JK \), \( KL \), and \( JL \). The angles opposite these sides are \( \angle L \), \( \angle J \), and \( \angle K \), respectively.

So, by the Triangle Longer Side Theorem, the angles from smallest to largest are \( \angle L \), \( \angle J \), and \( \angle K \).

EXAMPLE 4  Ordering Side Lengths of a Triangle

List the sides of \( \triangle DEF \) in order from shortest to longest.

**SOLUTION**

First, find \( m\angle F \) using the Triangle Sum Theorem.

\[
m\angle D + m\angle E + m\angle F = 180^\circ
\]

\[
51^\circ + 47^\circ + m\angle F = 180^\circ
\]

\[
m\angle F = 82^\circ
\]

The angles from smallest to largest are \( \angle E \), \( \angle D \), and \( \angle F \). The sides opposite these angles are \( DF \), \( EF \), and \( DE \), respectively.

So, by the Triangle Larger Angle Theorem, the sides from shortest to longest are \( DF \), \( EF \), and \( DE \).

Monitoring Progress

2. List the angles of \( \triangle PQR \) in order from smallest to largest.

3. List the sides of \( \triangle RST \) in order from shortest to longest.
Finding Possible Side Lengths

A triangle has one side of length 14 and another side of length 9. Describe the possible lengths of the third side.

**SOLUTION**

Let $x$ represent the length of the third side. Draw diagrams to help visualize the small and large values of $x$. Then use the Triangle Inequality Theorem to write and solve inequalities.

**Small values of $x$**

$$ x + 9 > 14 $$

$$ x > 5 $$

**Large values of $x$**

$$ 9 + 14 > x $$

$$ 23 > x, \text{ or } x < 23 $$

The length of the third side must be greater than 5 and less than 23.

**EXAMPLE 5** Finding Possible Side Lengths

A triangle has one side of length 14 and another side of length 9. Describe the possible lengths of the third side.

**SOLUTION**

Let $x$ represent the length of the third side. Draw diagrams to help visualize the small and large values of $x$. Then use the Triangle Inequality Theorem to write and solve inequalities.

**Small values of $x$**

$$ x + 9 > 14 $$

$$ x > 5 $$

**Large values of $x$**

$$ 9 + 14 > x $$

$$ 23 > x, \text{ or } x < 23 $$

The length of the third side must be greater than 5 and less than 23.

**Monitoring Progress**

4. A triangle has one side of length 12 inches and another side of length 20 inches. Describe the possible lengths of the third side.

Decide whether it is possible to construct a triangle with the given side lengths. Explain your reasoning.

5. 4 ft, 9 ft, 10 ft

6. 8 m, 9 m, 18 m

7. 5 cm, 7 cm, 12 cm
1. **VOCABULARY** Why is an indirect proof also called *proof by contradiction*?

2. **WRITING** How can you tell which side of a triangle is the longest from the angle measures of the triangle? How can you tell which side is the shortest?

**Monitoring Progress and Modeling with Mathematics**

In Exercises 3–6, write the first step in an indirect proof of the statement. *(See Example 1.)*

3. If \( WV + VU \neq 12 \) inches and \( VU = 5 \) inches, then \( WV \neq 7 \) inches.

4. If \( x \) and \( y \) are odd integers, then \( xy \) is odd.

5. In \( \triangle ABC \), if \( m\angle A = 100^\circ \), then \( \angle B \) is not a right angle.

6. In \( \triangle JKL \), if \( M \) is the midpoint of \( \overline{KL} \), then \( \overline{JM} \) is a median.

In Exercises 7 and 8, determine which two statements contradict each other. Explain your reasoning.

7. (A) \( \triangle LMN \) is a right triangle.
   (B) \( \angle L \equiv \angle N \)
   (C) \( \triangle LMN \) is equilateral.

8. (A) Both \( \angle X \) and \( \angle Y \) have measures greater than 20°.
   (B) Both \( \angle X \) and \( \angle Y \) have measures less than 30°.
   (C) \( m\angle X + m\angle Y = 62^\circ \)

In Exercises 9 and 10, use a ruler and protractor to draw the given type of triangle. Mark the largest angle and longest side in red and the smallest angle and shortest side in blue. What do you notice? *(See Example 2.)*

9. acute scalene
10. right scalene

In Exercises 11 and 12, list the angles of the given triangle from smallest to largest. *(See Example 3.)*

11. \( R \)
    \( 10 \)
    \( 9 \)

12. \( J \)
    \( 28 \)
    \( 13 \)

In Exercises 13–16, list the sides of the given triangle from shortest to longest. *(See Example 4.)*

13.

14.

15.

16.

In Exercises 17–20, describe the possible lengths of the third side of the triangle given the lengths of the other two sides. *(See Example 5.)*

17. 5 inches, 12 inches
18. 12 feet, 18 feet
19. 2 feet, 40 inches
20. 25 meters, 25 meters

In Exercises 21–24, is it possible to construct a triangle with the given side lengths? If not, explain why not.

21. 6, 7, 11
22. 3, 6, 9
23. 28, 17, 46
24. 35, 120, 125

25. **ERROR ANALYSIS** Describe and correct the error in writing the first step of an indirect proof.

---

**Show that \( \angle A \) is obtuse.**

**Step 1** Assume temporarily that \( \angle A \) is acute.
26. **ERROR ANALYSIS** Describe and correct the error in labeling the side lengths 1, 2, and \( \sqrt{3} \) on the triangle.

27. **REASONING** You are a lawyer representing a client who has been accused of a crime. The crime took place in Los Angeles, California. Security footage shows your client in New York at the time of the crime. Explain how to use indirect reasoning to prove your client is innocent.

28. **REASONING** Your class has fewer than 30 students. The teacher divides your class into two groups. The first group has 15 students. Use indirect reasoning to show that the second group must have fewer than 15 students.

29. **PROBLEM SOLVING** Which statement about \( \triangle TUV \) is false?
   - (A) \( UV > TU \)
   - (B) \( UV + TV > TU \)
   - (C) \( UV < TV \)
   - (D) \( \triangle TUV \) is isosceles.

30. **PROBLEM SOLVING** In \( \triangle RST \), which is a possible side length for \( ST \)? Select all that apply.
   - (A) 7
   - (B) 8
   - (C) 9
   - (D) 10

31. **PROOF** Write an indirect proof that an odd number is not divisible by 4.

32. **PROOF** Write an indirect proof of the statement “In \( \triangle QRS \), if \( m\angle Q + m\angle R = 90^\circ \), then \( m\angle S = 90^\circ \).”

33. **WRITING** Explain why the hypotenuse of a right triangle must always be longer than either leg.

34. **CRITICAL THINKING** Is it possible to decide if three side lengths form a triangle without checking all three inequalities shown in the Triangle Inequality Theorem? Explain your reasoning.

35. **MODELING WITH MATHEMATICS** You can estimate the width of the river from point A to the tree at point B by measuring the angle to the tree at several locations along the riverbank. The diagram shows the results for locations C and D.

   a. Using \( \triangle BCA \) and \( \triangle BDA \), determine the possible widths of the river. Explain your reasoning.

   b. What could you do if you wanted a closer estimate?

36. **MODELING WITH MATHEMATICS** You travel from Fort Peck Lake to Glacier National Park and from Glacier National Park to Granite Peak.

   a. Write two inequalities to represent the possible distances from Granite Peak back to Fort Peck Lake.

   b. How is your answer to part (a) affected if you know that \( m\angle 2 < m\angle 1 \) and \( m\angle 2 < m\angle 3 \)?

37. **REASONING** In the figure, \( \overline{XY} \) bisects \( \angle WYZ \). List all six angles of \( \triangle XYZ \) and \( \triangle WXY \) in order from smallest to largest. Explain your reasoning.

38. **MATHEMATICAL CONNECTIONS** In \( \triangle DEF \), \( m\angle D = (x + 25)^\circ \), \( m\angle E = (2x - 4)^\circ \), and \( m\angle F = 63^\circ \). List the side lengths and angle measures of the triangle in order from least to greatest.
39. **ANALYZING RELATIONSHIPS** Another triangle inequality relationship is given by the Exterior Angle Inequality Theorem. It states:

The measure of an exterior angle of a triangle is greater than the measure of either of the nonadjacent interior angles.

Explain how you know that \( m\angle 1 > m\angle A \) and \( m\angle 1 > m\angle B \) in \( \triangle ABC \) with exterior angle \( \angle 1 \).

**MATHEMATICAL CONNECTIONS** In Exercises 40 and 41, describe the possible values of \( x \).

40. \( 5x - 9 \) \( 2x + 10 \)
41. \( 2x + 3 \) \( 3x - 1 \)

42. **HOW DO YOU SEE IT?** Your house is on the corner of Hill Street and Eighth Street. The library is on the corner of View Street and Seventh Street. What is the shortest route to get from your house to the library? Explain your reasoning.

43. **PROVING A THEOREM** Use the diagram to prove the Triangle Longer Side Theorem.

Given \( BC > AB, BD = BA \)

Prove \( m\angle BAC > m\angle C \)

44. **USING STRUCTURE** The length of the base of an isosceles triangle is \( \ell \). Describe the possible lengths for each leg. Explain your reasoning.

45. **MAKING AN ARGUMENT** Your classmate claims to have drawn a triangle with one side length of 13 inches and a perimeter of 2 feet. Is this possible? Explain your reasoning.

46. **THOUGHT PROVOKING** Cut two pieces of string that are each 24 centimeters long. Construct an isosceles triangle out of one string and a scalene triangle out of the other. Measure and record the side lengths. Then classify each triangle by its angles.

47. **PROVING A THEOREM** Prove the Triangle Inequality Theorem.

Given \( \triangle ABC \)

Prove \( AB + BC > AC, AC + BC > AB, \) and \( AB + AC > BC \)

48. **ATTENDING TO PRECISION** The perimeter of \( \triangle HGF \) must be between what two integers? Explain your reasoning.

49. **PROOF** Write an indirect proof that a perpendicular segment is the shortest segment from a point to a plane.

Given \( PC \perp \text{plane } M \)

Prove \( PC \) is the shortest segment from \( P \) to plane \( M \).

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Name the included angle between the pair of sides given. (Skills Review Handbook)

50. \( \overline{AE} \) and \( \overline{BE} \)
51. \( \overline{AC} \) and \( \overline{DC} \)
52. \( \overline{AD} \) and \( \overline{DC} \)
53. \( \overline{CE} \) and \( \overline{BE} \)
**6.7 Inequalities in Two Triangles**

**Essential Question** If two sides of one triangle are congruent to two sides of another triangle, what can you say about the third sides of the triangles?

**EXPLORATION 1** Comparing Measures in Triangles

Work with a partner. Use dynamic geometry software.

a. Draw \( \triangle ABC \), as shown below.

b. Draw the circle with center \( C(3, 3) \) through the point \( A(1, 3) \).

c. Draw \( \triangle DBC \) so that \( D \) is a point on the circle.

\[ D(4.75, 2.03) \]

\[ AC = 2 \]

\[ BC = 3 \]

\[ DC = 2 \]

\[ AB = 3.61 \]

\[ DB = 2.68 \]

\[ m\angle ACB \]

\[ m\angle BCD \]

---

d. Which two sides of \( \triangle ABC \) are congruent to two sides of \( \triangle DBC \)? Justify your answer.

e. Compare the lengths of \( AB \) and \( DB \). Then compare the measures of \( m\angle ACB \) and \( m\angle DCB \). Are the results what you expected? Explain.

f. Drag point \( D \) to several locations on the circle. At each location, repeat part (e). Copy and record your results in the table below.

<table>
<thead>
<tr>
<th>( D )</th>
<th>( AC )</th>
<th>( BC )</th>
<th>( AB )</th>
<th>( BD )</th>
<th>( m\angle ACB )</th>
<th>( m\angle BCD )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ((4.75, 2.03))</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

g. Look for a pattern of the measures in your table. Then write a conjecture that summarizes your observations.

**Communicate Your Answer**

2. If two sides of one triangle are congruent to two sides of another triangle, what can you say about the third sides of the triangles?

3. Explain how you can use the hinge shown at the left to model the concept described in Question 2.
What You Will Learn

- Compare measures in triangles.
- Solve real-life problems using the Hinge Theorem.

Comparing Measures in Triangles

Imagine a gate between fence posts $A$ and $B$ that has hinges at $A$ and swings open at $B$.

As the gate swings open, you can think of $\triangle ABC$, with side $\overline{AC}$ formed by the gate itself, side $\overline{AB}$ representing the distance between the fence posts, and side $\overline{BC}$ representing the opening between post $B$ and the outer edge of the gate.

Notice that as the gate opens wider, both the measure of $\angle A$ and the distance $BC$ increase. This suggests the Hinge Theorem.

Theorems

Hinge Theorem

If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first is longer than the third side of the second.

Proof [BigIdeasMath.com]

Converse of the Hinge Theorem

If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second, then the included angle of the first is larger than the included angle of the second.

Proof Example 3, p. 387

Example 1

Using the Converse of the Hinge Theorem

Given that $\overline{ST} \cong \overline{PR}$, how does $m\angle PST$ compare to $m\angle SPR$?

SOLUTION

You are given that $\overline{ST} \cong \overline{PR}$, and you know that $\overline{PS} \cong \overline{PS}$ by the Reflexive Property of Congruence. Because $24$ inches $> 23$ inches, $PT > SR$. So, two sides of $\triangle STP$ are congruent to two sides of $\triangle PRS$ and the third side of $\triangle STP$ is longer.

By the Converse of the Hinge Theorem, $m\angle PST > m\angle SPR$. 

Copyright © Big Ideas Learning, LLC. All rights reserved.
Using the Hinge Theorem

Given that $\overline{JK} \cong \overline{LK}$, how does $JM$ compare to $LM$?

**SOLUTION**

You are given that $\overline{JK} \cong \overline{LK}$, and you know that $\overline{KM} \cong \overline{KM}$ by the Reflexive Property of Congruence. Because $64^\circ > 61^\circ$, $m\angle JKM > m\angle LKM$. So, two sides of $\triangle JKM$ are congruent to two sides of $\triangle LKM$, and the included angle in $\triangle JKM$ is larger.

By the Hinge Theorem, $JM > LM$.

**Monitoring Progress**

1. If $PR = PS$ and $m\angle QPR > m\angle QPS$, which is longer, $\overline{SQ}$ or $\overline{RQ}$?

2. If $PR = PS$ and $RQ < SQ$, which is larger, $\angle RPQ$ or $\angle SPQ$?

**EXAMPLE 3** Proving the Converse of the Hinge Theorem

Write an indirect proof of the Converse of the Hinge Theorem.

Given $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}, AC > DF$

Prove $m\angle B > m\angle E$

**Indirect Proof**

**Step 1** Assume temporarily that $m\angle B \geq m\angle E$. Then it follows that either $m\angle B < m\angle E$ or $m\angle B = m\angle E$.

**Step 2** If $m\angle B < m\angle E$, then $AC < DF$ by the Hinge Theorem.

If $m\angle B = m\angle E$, then $\triangle ABC \cong \triangle DEF$ by the SAS Congruence Theorem and $AC = DF$.

**Step 3** Both conclusions contradict the given statement that $AC > DF$. So, the temporary assumption that $m\angle B \geq m\angle E$ cannot be true. This proves that $m\angle B > m\angle E$.

**EXAMPLE 4** Proving Triangle Relationships

Write a paragraph proof.

Given $\angle XWY \cong \angle XYW, WZ > YZ$

Prove $m\angle WXZ > m\angle YXZ$

**Paragraph Proof** Because $\angle XWY \cong \angle XYW, \overline{XY} \cong \overline{XY}$ by the Converse of the Base Angles Theorem. By the Reflexive Property of Congruence, $\overline{XZ} \cong \overline{XZ}$. Because $WZ > YZ, m\angle WXZ > m\angle YXZ$ by the Converse of the Hinge Theorem.

**Monitoring Progress**

3. Write a temporary assumption you can make to prove the Hinge Theorem indirectly. What two cases does that assumption lead to?
Solving Real-Life Problems

**EXAMPLE 5** Solving a Real-Life Problem

Two groups of bikers leave the same camp heading in opposite directions. Each group travels 2 miles, then changes direction and travels 1.2 miles. Group A starts due east and then turns 45° toward north. Group B starts due west and then turns 30° toward south. Which group is farther from camp? Explain your reasoning.

**SOLUTION**

1. **Understand the Problem** You know the distances and directions that the groups of bikers travel. You need to determine which group is farther from camp. You can interpret a turn of 45° toward north, as shown.

2. **Make a Plan** Draw a diagram that represents the situation and mark the given measures. The distances that the groups bike and the distances back to camp form two triangles. The triangles have two congruent side lengths of 2 miles and 1.2 miles. Include the third side of each triangle in the diagram.

3. **Solve the Problem** Use linear pairs to find the included angles for the paths that the groups take.

   **Group A:** \(180° - 45° = 135°\)  
   **Group B:** \(180° - 30° = 150°\)

   The included angles are 135° and 150°.

4. **Look Back** Because 150° > 135°, the distance Group B is from camp is greater than the distance Group A is from camp by the Hinge Theorem.

   So, Group B is farther from camp.

4. **WHAT IF?** In Example 5, Group C leaves camp and travels 2 miles due north, then turns 40° toward east and travels 1.2 miles. Compare the distances from camp for all three groups.

---

Monitoring Progress HELP IN ENGLISH AND SPANISH AT BigIdeasMath.com

4. **WHAT IF?** In Example 5, Group C leaves camp and travels 2 miles due north, then turns 40° toward east and travels 1.2 miles. Compare the distances from camp for all three groups.
In Exercises 3–6, copy and complete the statement with <, >, or =. Explain your reasoning. (See Example 1.)

3. \( m\angle 1 \) _____ \( m\angle 2 \)

4. \( m\angle 1 \) _____ \( m\angle 2 \)

5. \( m\angle 1 \) _____ \( m\angle 2 \)

6. \( m\angle 1 \) _____ \( m\angle 2 \)

In Exercises 7–10, copy and complete the statement with <, >, or =. Explain your reasoning. (See Example 2.)

7. \( AD \) _____ \( CD \)

8. \( MN \) _____ \( LK \)

9. \( TR \) _____ \( UR \)

10. \( AC \) _____ \( DC \)

PROOF In Exercises 11 and 12, write a proof. (See Example 4.)

11. Given \( \overline{XY} \equiv \overline{YZ} \), \( m\angle WYZ > m\angle WYX \) 

   Prove \( WZ > WX \)

12. Given \( \overline{BC} \equiv \overline{DA} \), \( DC < AB \) 

   Prove \( m\angle BCA > m\angle DAC \)

In Exercises 13 and 14, you and your friend leave on different flights from the same airport. Determine which flight is farther from the airport. Explain your reasoning. (See Example 5.)

13. Your flight: Flies 100 miles due west, then turns 20° toward north and flies 50 miles.

   Friend’s flight: Flies 100 miles due north, then turns 30° toward east and flies 50 miles.

14. Your flight: Flies 210 miles due south, then turns 70° toward west and flies 80 miles.

   Friend’s flight: Flies 80 miles due north, then turns 50° toward east and flies 210 miles.
15. **ERROR ANALYSIS** Describe and correct the error in using the Hinge Theorem.

By the Hinge Theorem, \( PQ < SR \).

16. **REPEATED REASONING** Which is a possible measure for \( \angle JKM \)? Select all that apply.

- A) 15°  
- B) 22°  
- C) 25°  
- D) 35°

17. **DRAWING CONCLUSIONS** The path from \( E \) to \( F \) is longer than the path from \( E \) to \( D \). The path from \( G \) to \( D \) is the same length as the path from \( G \) to \( F \). What can you conclude about the angles of the paths? Explain your reasoning.

18. **ABSTRACT REASONING** In \( \triangle EFG \), the bisector of \( \angle F \) intersects the bisector of \( \angle G \) at point \( H \). Explain why \( FH \) must be longer than \( FH \) or \( HG \).

19. **ABSTRACT REASONING** \( \overline{NR} \) is a median of \( \triangle NPQ \), and \( NQ > NP \). Explain why \( \angle NRQ \) is obtuse.

20. **MATHEMATICAL CONNECTIONS** In Exercises 20 and 21, write and solve an inequality for the possible values of \( x \).

21. **MATHEMATICAL CONNECTIONS** In Exercises 20 and 21, write and solve an inequality for the possible values of \( x \).

22. **HOW DO YOU SEE IT?** In the diagram, triangles are formed by the locations of the players on the basketball court. The dashed lines represent the possible paths of the basketball as the players pass. How does \( \angle ACB \) compare with \( \angle ACD \)?

23. **CRITICAL THINKING** In \( \triangle ABC \), the altitudes from \( B \) and \( C \) meet at point \( D \), and \( \angle BAC > \angle BDC \). What is true about \( \triangle ABC \)? Justify your answer.

24. **THOUGHT PROVOKING** The postulates and theorems in this book represent Euclidean geometry. In spherical geometry, all points are on the surface of a sphere. A line is a circle on the sphere whose diameter is equal to the diameter of the sphere. In spherical geometry, state an inequality involving the sum of the angles of a triangle. Find a formula for the area of a triangle in spherical geometry.

---

**Maintaining Mathematical Proficiency**

Find the value of \( x \).  

25. \( \triangle ABC \) with \( \angle B = 27° \) and \( \angle C = 115° \). Find \( \angle A \).

26. \( \triangle ABC \) with \( \angle A = 36° \) and \( \angle B = 124° \). Find \( \angle C \).

27. \( \triangle ABC \) with \( \angle B = 44° \) and \( \angle C = 64° \). Find \( \angle A \).

28. \( \triangle ABC \) with \( \angle A = x° \) and \( \angle C = (x + 10)° \). Find \( \angle B \).

---

Copyright © Big Ideas Learning, LLC. All rights reserved.
6.5–6.7 What Did You Learn?

Core Vocabulary

midsegment of a triangle, p. 372
indirect proof, p. 378

Core Concepts

Section 6.5
Using the Midsegment of a Triangle, p. 372
Triangle Midsegment Theorem, p. 373

Section 6.6
How to Write an Indirect Proof (Proof by Contradiction), p. 378
Triangle Longer Side Theorem, p. 379
Triangle Larger Angle Theorem, p. 379
Triangle Inequality Theorem, p. 381

Section 6.7
Hinge Theorem, p. 386
Converse of the Hinge Theorem, p. 386

Mathematical Practices

1. In Exercise 25 on page 376, analyze the relationship between the stage and the total perimeter of all the shaded triangles at that stage. Then predict the total perimeter of all the shaded triangles in Stage 4.

2. In Exercise 17 on page 382, write all three inequalities using the Triangle Inequality Theorem. Determine the reasonableness of each one. Why do you only need to use two of the three inequalities?

3. In Exercise 23 on page 390, try all three cases of triangles (acute, right, obtuse) to gain insight into the solution.

Performance Task:

Building a Roof Truss

The simple roof truss is also called a planar truss because all its components lie in a two-dimensional plane. How can this structure be extended to three-dimensional space? What applications would this type of structure be used for?

To explore the answers to these questions and more, check out the Performance Task and Real-Life STEM video at BigIdeasMath.com.
### 6.1 Proving Geometric Relationships (pp. 335–342)

**a. Rewrite the two-column proof into a paragraph proof.**

**Given** \( \angle 6 \cong \angle 7 \)

**Prove** \( \angle 7 \cong \angle 2 \)

**Two-Column Proof**

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle 6 \cong \angle 7 )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \angle 6 \cong \angle 2 )</td>
<td>2. Vertical Angles Congruence Theorem</td>
</tr>
<tr>
<td>3. ( \angle 7 \cong \angle 2 )</td>
<td>3. Transitive Property of Congruence</td>
</tr>
</tbody>
</table>

**Paragraph Proof**

\( \angle 6 \) and \( \angle 7 \) are congruent. By the Vertical Angles Congruence Theorem, \( \angle 6 \cong \angle 2 \). So, by the Transitive Property of Congruence, \( \angle 7 \cong \angle 2 \).

**b. Write a coordinate proof.**

**Given** Coordinates of vertices of \( \triangle OAC \) and \( \triangle BAC \)

**Prove** \( \triangle OAC \cong \triangle BAC \)

Segments \( \overline{OA} \) and \( \overline{BA} \) have the same length, so \( \overline{OA} \equiv \overline{BA} \).

\[
OA = \sqrt{(c - 2)^2} = \sqrt{c^2 - 4c + 4} = \sqrt{c^2 - 4c + 4}
\]

\[
BA = \sqrt{(c - c)^2} = \sqrt{0} = 0
\]

Segments \( \overline{OC} \) and \( \overline{BC} \) have the same length, so \( \overline{OC} \equiv \overline{BC} \).

\[
OC = \sqrt{(0 - c)^2} = \sqrt{c^2} = \sqrt{c^2}
\]

\[
BC = \sqrt{(c - c)^2} = \sqrt{0} = 0
\]

By the Reflexive Property of Congruence, \( \overline{AC} \equiv \overline{AC} \).

So, you can apply the SSS Congruence Theorem to conclude that \( \triangle OAC \cong \triangle BAC \).

1. **Use the figure in part (a). Write a proof using any format.**

   **Given** \( \angle 5 \) and \( \angle 6 \) are complementary.

   \[ m\angle 5 + m\angle 7 = 90^\circ \]

   **Prove** \( \angle 6 \cong \angle 7 \)

2. **Use the figure at the right. Write a coordinate proof.**

   **Given** Coordinates of vertices of \( \triangle OBC \)

   **Prove** \( \triangle OBC \) is isosceles.
6.2 Perpendicular and Angle Bisectors (pp. 343–350)

Find each measure.

a. $AD$
   
   From the figure, $\overline{AC}$ is the perpendicular bisector of $\overline{BD}$.
   
   $AB = AD$ \hspace{1cm} \text{Perpendicular Bisector Theorem}
   
   $4x + 3 = 6x - 9$ \hspace{1cm} \text{Substitute.}
   
   $x = 6$ \hspace{1cm} \text{Solve for } x.
   
   $\blacktriangleright$ So, $AD = 6(6) - 9 = 27$.

b. $FG$
   
   Because $\overline{EH} = \overline{GH}$ and $\overline{HF} \perp \overline{EG}$, $\overline{HF}$ is the perpendicular bisector of $\overline{EG}$ by the Converse of the Perpendicular Bisector Theorem. By the definition of segment bisector, $\overline{EF} = \overline{FG}$.
   
   $\blacktriangleright$ So, $FG = EF = 6$.

c. $LM$
   
   $JM = LM$ \hspace{1cm} \text{Angle Bisector Theorem}
   
   $8x - 15 = 3x$ \hspace{1cm} \text{Substitute.}
   
   $x = 3$ \hspace{1cm} \text{Solve for } x.
   
   $\blacktriangleright$ So, $LM = 3x = 3(3) = 9$.

d. $m\angle XWZ$
   
   Because $\overline{ZX} \perp \overline{WX}$ and $\overline{ZY} \perp \overline{WY}$ and $ZX = ZY = 8$, $\overline{WZ}$ bisects $\angle XWY$ by the Converse of the Angle Bisector Theorem.
   
   $\blacktriangleright$ So, $m\angle XWZ = m\angle YWZ = 46^\circ$.

Find the indicated measure. Explain your reasoning.

3. $DC$
   
   $20$ \hspace{1cm} $7x - 15$ \hspace{1cm} $8x + 3$

4. $RS$
   
   $9x - 4$ \hspace{1cm} $6x + 5$

5. $m\angle JFH$
   
   $47^\circ$ \hspace{1cm} $9$ \hspace{1cm} $9$
a. Find the coordinates of the circumcenter of \( \triangle QRS \) with vertices \( Q(3, 3) \), \( R(5, 7) \), and \( S(9, 3) \).

**Step 1** Graph \( \triangle QRS \).

**Step 2** Find equations for two perpendicular bisectors.

The midpoint of \( \overline{QS} \) is (6, 3). The line through (6, 3) that is perpendicular to \( \overline{QS} \) is \( x = 6 \).

The midpoint of \( \overline{QR} \) is (4, 5). The line through (4, 5) that is perpendicular to \( \overline{QR} \) is \( y = -\frac{1}{2}x + 7 \).

**Step 3** Find the point where \( x = 6 \) and \( y = -\frac{1}{2}x + 7 \) intersect. They intersect at (6, 4).

![Graph showing the circumcenter of \( \triangle QRS \)](image)

So, the coordinates of the circumcenter are (6, 4).

b. Point \( N \) is the incenter of \( \triangle ABC \). In the figure shown, \( ND = 4x + 7 \) and \( NE = 3x + 9 \). Find \( NF \).

**Step 1** Solve for \( x \).

\[ ND = NE \quad \text{Incenter Theorem} \]
\[ 4x + 7 = 3x + 9 \quad \text{Substitute} \]
\[ x = 2 \quad \text{Solve for } x \]

**Step 2** Find \( ND \) (or \( NE \)).

\[ ND = 4x + 7 = 4(2) + 7 = 15 \]

So, because \( ND = NF \), \( NF = 15 \).

Find the coordinates of the circumcenter of the triangle with the given vertices.

6. \( T(-6, -5), U(0, -1), V(0, -5) \)

7. \( X(-2, 1), Y(2, -3), Z(6, -3) \)

8. Point \( D \) is the incenter of \( \triangle LMN \). Find the value of \( x \).
6.4 Medians and Altitudes of Triangles  (pp. 361–368)

Find the coordinates of the centroid of $\triangle TUV$ with vertices $T(1, -8)$, $U(4, -1)$, and $V(7, -6)$.

**Step 1** Graph $\triangle TUV$.

**Step 2** Use the Midpoint Formula to find the midpoint $W$ of $\overline{TV}$.

\[
W\left(\frac{1 + 7}{2}, \frac{-8 + (-6)}{2}\right) = (4, -7)
\]

**Step 3** Find the centroid. It is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The distance from vertex $U(4, -1)$ to $W(4, -7)$ is $-1 - (-7) = 6$ units.

So, the centroid is $\frac{2}{3}(6) = 4$ units down from vertex $U$ on $\overline{UW}$.

So, the coordinates of the centroid $P$ are $(4, -1 - 4)$, or $(4, -5)$.

Find the coordinates of the centroid of the triangle with the given vertices.

9. $A(-10, 3), B(-4, 5), C(-4, 1)$

10. $D(2, -8), E(2, -2), F(8, -2)$

Tell whether the orthocenter of the triangle with the given vertices is inside, on, or outside the triangle. Then find the coordinates of the orthocenter.

11. $G(1, 6), H(5, 6), J(3, 1)$

12. $K(-8, 5), L(-6, 3), M(0, 5)$

6.5 The Triangle Midsegment Theorem  (pp. 371–376)

In $\triangle JKL$, show that midsegment $\overline{MN}$ is parallel to $\overline{JL}$ and that $MN = \frac{1}{2} JL$.

**Step 1** Find the coordinates of $M$ and $N$ by finding the midpoints of $\overline{JK}$ and $\overline{KL}$.

\[
M\left(\frac{-8 + (-4)}{2}, \frac{1 + 7}{2}\right) = M\left(-\frac{12}{2}, \frac{8}{2}\right) = M(-6, 4)
\]

\[
N\left(\frac{-4 + (-2)}{2}, \frac{7 + 3}{2}\right) = N\left(-\frac{6}{2}, \frac{10}{2}\right) = N(-3, 5)
\]

**Step 2** Find and compare the slopes of $\overline{MN}$ and $\overline{JL}$.

slope of $\overline{MN} = \frac{-3 - (-6)}{2} = \frac{1}{3}$

slope of $\overline{JL} = \frac{3 - 1}{2 - (-8)} = \frac{2}{10} = \frac{1}{5}$

Because the slopes are the same, $\overline{MN}$ is parallel to $\overline{JL}$.

**Step 3** Find and compare the lengths of $\overline{MN}$ and $\overline{JL}$.

\[
MN = \sqrt{[-3 - (-6)]^2 + (5 - 4)^2} = \sqrt{9 + 1} = \sqrt{10}
\]

\[
JL = \sqrt{[-2 - (-8)]^2 + (3 - 1)^2} = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10}
\]

Because $\sqrt{10} = \frac{1}{2}(2\sqrt{10})$, $MN = \frac{1}{2} JL$.

Find the coordinates of the vertices of the midsegment triangle for the triangle with the given vertices.

13. $A(-6, 8), B(-6, 4), C(0, 4)$

14. $D(-3, 1), E(3, 5), F(1, -5)$
6.6 Indirect Proof and Inequalities in One Triangle (pp. 377–384)

a. List the sides of \( \triangle ABC \) in order from shortest to longest.

First, find \( m\angle C \) using the Triangle Sum Theorem.

\[
35^\circ + 95^\circ + m\angle C = 180^\circ \\
m\angle C = 50^\circ
\]

The angles from smallest to largest are \( \angle A, \angle C, \) and \( \angle B \). The sides opposite these angles are \( BC, AB, \) and \( AC \), respectively.

\[ \rightarrow \text{So, by the Triangle Larger Angle Theorem, the sides from shortest to longest are } BC, AB, \text{ and } AC. \]

b. List the angles of \( \triangle DEF \) in order from smallest to largest.

The sides from shortest to longest are \( DF, EF, \) and \( DE \). The angles opposite these sides are \( \angle E, \angle D, \) and \( \angle F \), respectively.

\[ \rightarrow \text{So, by the Triangle Longer Side Theorem, the angles from smallest to largest are } \angle E, \angle D, \text{ and } \angle F. \]

Describe the possible lengths of the third side of the triangle given the lengths of the other two sides.

15. 4 inches, 8 inches  
16. 6 meters, 9 meters  
17. 11 feet, 18 feet  
18. Write an indirect proof of the statement “In \( \triangle XYZ \), if \( XY = 4 \) and \( XZ = 8 \), then \( YZ > 4 \).”

6.7 Inequalities in Two Triangles (pp. 385–390)

Given that \( WZ \cong YZ \), how does \( XY \) compare to \( XW \)?

You are given that \( WZ \cong YZ \), and you know that \( XZ \cong XZ \) by the Reflexive Property of Congruence.

Because \( 90^\circ > 80^\circ, m\angle XZY > m\angle XZW \). So, two sides of \( \triangle XZY \) are congruent to two sides of \( \triangle XZW \) and the included angle in \( \triangle XZY \) is larger.

\[ \rightarrow \text{By the Hinge Theorem, } XY > XW. \]

Use the diagram.

19. If \( RQ = RS \) and \( m\angle QRT > m\angle SRT \), then how does \( QT \) compare to \( ST \)?

20. If \( RQ = RS \) and \( QT > ST \), then how does \( \angle QRT \) compare to \( \angle SRT \)?
In Exercises 1 and 2, \( MN \) is a midsegment of \( \triangle JKL \). Find the value of \( x \).

1. \[
\begin{align*}
M & \quad x \quad 12 \\
J & \quad N \quad L \\
K &
\end{align*}
\]

2. \[
\begin{align*}
J & \quad 9 \quad M \quad x \quad L \\
N & \quad 8.5 \\
K & \quad 17 \\
\end{align*}
\]

Find the indicated measure. Identify the theorem you use.

3. \( ST \)

4. \( WY \)

5. \( BW \)

Copy and complete the statement with <, >, or =.

6. \( AB \) __ \( CB \)

7. \( m\angle 1 \) ___ \( m\angle 2 \)

8. \( m\angle MNP \) ___ \( m\angle NPM \)

9. Find the coordinates of the circumcenter, orthocenter, and centroid of the triangle with vertices \( A(0, -2) \), \( B(4, -2) \), and \( C(0, 6) \).

10. Write an indirect proof of the Corollary to the Base Angles Theorem:
If \( \triangle PQR \) is equilateral, then it is equiangular.

11. \( \triangle DEF \) is a right triangle with area \( A \). Use the area for \( \triangle DEF \) to write an expression for the area of \( \triangle GEH \). Justify your answer.

12. Prove the Corollary to the Triangle Sum Theorem using any format.

   Given \( \triangle ABC \) is a right triangle.

   Prove \( \angle A \) and \( \angle B \) are complementary.

In Exercises 13–15, use the map.

13. Describe the possible lengths of Pine Avenue.

14. You ride your bike along a trail that represents the shortest distance from the beach to Main Street. You end up exactly halfway between your house and the movie theatre. How long is Pine Avenue? Explain.

15. A market is the same distance from your house, the movie theater, and the beach. Copy the map and locate the market.
1. Which definition(s) and/or theorem(s) do you need to use to prove the Converse of the Perpendicular Bisector Theorem? Select all that apply.

**Given** \( CA = CB \)

**Prove** Point \( C \) lies on the perpendicular bisector of \( AB \).

- definition of perpendicular bisector
- definition of angle bisector
- definition of segment congruence
- definition of angle congruence
- Base Angles Theorem
- Converse of the Base Angles Theorem
- ASA Congruence Theorem
- AAS Congruence Theorem

2. Which of the following values are \( x \)-coordinates of the solutions of the system?

\[
\begin{align*}
y &= x^2 - 3x + 3 \\
y &= 3x - 5
\end{align*}
\]

\[
\begin{array}{cccc}
-7 & -4 & -3 & -2 & -1 \\
1 & 2 & 3 & 4 & 7
\end{array}
\]

3. What are the coordinates of the centroid of \( \triangle LMN \)?

- A) \((2, 5)\)
- B) \((3, 5)\)
- C) \((4, 5)\)
- D) \((5, 5)\)

4. Use the steps in the construction to explain how you know that the circle is circumscribed about \( \triangle ABC \).

**Step 1**

**Step 2**

**Step 3**
5. According to a survey, 58% of voting-age citizens in a town are planning to vote in an upcoming election. You randomly select 10 citizens to survey.
   a. What is the probability that less than 4 citizens are planning to vote?
   b. What is the probability that exactly 4 citizens are planning to vote?
   c. What is the probability that more than 4 citizens are planning to vote?

6. What are the solutions of \(2x^2 + 18 = -6\)?
   \(\text{A} \) \(i\sqrt{6} \text{ and } -i\sqrt{6}\)
   \(\text{B} \) \(2i\sqrt{3} \text{ and } -2i\sqrt{3}\)
   \(\text{C} \) \(i\sqrt{15} \text{ and } -i\sqrt{15}\)
   \(\text{D} \) \(-i\sqrt{21} \text{ and } i\sqrt{21}\)

7. Use the graph of \(\triangle QRS\).

   a. Find the coordinates of the vertices of the midsegment triangle. Label the vertices \(T\), \(U\), and \(V\).
   b. Show that each midsegment joining the midpoints of two sides is parallel to the third side and is equal to half the length of the third side.

8. The graph of which inequality is shown?
   \(\text{A} \) \(y > x^2 - 3x + 4\)
   \(\text{B} \) \(y \geq x^2 - 3x + 4\)
   \(\text{C} \) \(y < x^2 - 3x + 4\)
   \(\text{D} \) \(y \leq x^2 - 3x + 4\)
Chapter 6

Relationships Within Triangles
6 Relationships Within Triangles

6.1 Proving Geometric Relationships
6.2 Perpendicular and Angle Bisectors
6.3 Bisectors of Triangles
6.4 Medians and Altitudes of Triangles
6.5 The Triangle Midsegment Theorem
6.6 Indirect Proof and Inequalities in One Triangle
6.7 Inequalities in Two Triangles

SEE the Big Idea

Biking (p. 388)
Roof Truss (p. 373)
Bridge (p. 345)
Windmill (p. 360)
Montana (p. 383)

Chapter 6 Pacing Guide

<table>
<thead>
<tr>
<th>Section</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter Opener/ Mathematical Practices</td>
<td>0.5 Day</td>
</tr>
<tr>
<td>Section 1</td>
<td>0.5 Day</td>
</tr>
<tr>
<td>Section 2</td>
<td>2 Days</td>
</tr>
<tr>
<td>Section 3</td>
<td>2 Days</td>
</tr>
<tr>
<td>Section 4</td>
<td>2 Days</td>
</tr>
<tr>
<td>Quiz</td>
<td>0.5 Day</td>
</tr>
<tr>
<td>Section 5</td>
<td>1.5 Days</td>
</tr>
<tr>
<td>Section 6</td>
<td>2 Days</td>
</tr>
<tr>
<td>Section 7</td>
<td>1 Day</td>
</tr>
<tr>
<td>Chapter Review/ Chapter Tests</td>
<td>2 Days</td>
</tr>
<tr>
<td>Total Chapter 6</td>
<td>14 Days</td>
</tr>
<tr>
<td>Year-to-Date</td>
<td>92 Days</td>
</tr>
</tbody>
</table>
Chapter Summary

- This chapter uses the deductive skills developed in Math I to explore special segments in a triangle. These segments include perpendicular bisectors, angle bisectors, medians, altitudes, and midsegments.
- The first lesson serves as an introduction to proofs for students who did not cover this topic in Math I and as a review for students who did.
- Students are able to discover special properties of these segments by using dynamic geometry software. To prove these relationships, a variety of proof formats and approaches are used: transformational, synthetic, analytic, and paragraph.
- The last two lessons in the chapter are about inequalities within one triangle and in two triangles. The indirect proof is introduced and used to prove several of the theorems in these lessons.
- Triangles have been the geometric structure used to help students develop their deductive reasoning skills. In the next chapter, quadrilaterals and other polygons are studied.

What Your Students Have Learned

Middle School
- Construct geometric figures with given conditions.
- Draw polygons in the coordinate plane given the vertices and use the coordinates to find the lengths of the sides.
- Find the sum of the interior angles of a triangle and find the measure of the exterior angle of a triangle.
- Write and evaluate algebraic expressions.

Math I
- Measure and classify angles.
- Prove geometric relationships using two-column, paragraph, flowchart, and coordinate proofs.
- Prove triangle congruence by SAS, SSS, ASA, and AAS Congruence Theorems.
- Write and solve linear equations in one variable.

What Your Students Will Learn

Math II
- Use perpendicular bisectors and angle bisectors to find measures.
- Use and find the circumcenter, incenter, centroid, and orthocenter of a triangle.
- Use the Triangle Midsegment Theorem to find distances.
- Prove geometric relationships using indirect proofs.
- Relate the sides and angles of a triangle and use the Triangle Inequality theorem to find possible side lengths of a triangle.
- Use the Hinge Theorem to compare measures between two triangles.

Scaffolding in the Classroom

Graphic Organizers: Word Magnet
A Word Magnet can be used to organize information associated with a vocabulary word or term. Students write the word or term inside the magnet. Students write associated information on the blank lines that "radiate" from the magnet. Associated information can include, but is not limited to: other vocabulary words or terms, definitions, formulas, procedures, examples, and visuals. This type of organizer serves as a good summary tool because any information related to a topic can be included.
Maintaining Mathematical Proficiency

Writing an Equation of a Perpendicular Line
- An equation written in slope-intercept form, \( y = mx + b \), has a slope \( m \) and a \( y \)-intercept \( b \).
- Students should remember that in a coordinate plane, two nonvertical lines are perpendicular if and only if the product of their slopes is \(-1\).

**COMMON ERROR** Students substitute incorrectly when finding the value of \( b \). Have them write the slope-intercept form of an equation of a line, \( y = mx + b \). Then remind students to substitute the first coordinate of the given point for \( x \) and the second coordinate for \( y \).

Writing Compound Inequalities
- Be sure students understand the implications of the words **and** and **or**.
- Remind students that when a compound inequality uses the word **and**, the variable must satisfy both inequalities. When a compound inequality uses the word **or**, the variable needs to satisfy only one of the inequalities.

**COMMON ERROR** Students sometimes switch **and** and **or** when writing inequalities, which leads to confusion when they have to solve a compound inequality.

Mathematical Practices (continued on page 334)
- The *Mathematical Practices* page focuses attention on how mathematics is learned—process versus content. Page 334 demonstrates that by using technological tools, students can explore and discover geometric relationships. Although these relationships can be explored in other manners, dynamic geometry software can be very effective and efficient.
- Use the *Mathematical Practices* page to help students develop mathematical habits of mind—how mathematics can be explored and how mathematics is thought about.

Questioning in the Classroom

**What do you think?**
Allow students time to write their answers on individual whiteboards. After the allotted time, they can hold up their boards. The teacher can see what students are thinking and follow up with discussion.
Maintaining Mathematical Proficiency

Writing an Equation of a Perpendicular Line (Math I)

Example 1

Write the equation of a line passing through the point \((-2, 0)\) that is perpendicular to the line \(y = 2x + 8\).

Step 1

Find the slope \(m\) of the perpendicular line. The line \(y = 2x + 8\) has a slope of 2. Use the Slopes of Perpendicular Lines Theorem.

\[
2 \cdot m = -1
\]

The product of the slopes of \(\perp\) lines is \(-1\).

\[
m = -\frac{1}{2}
\]

Divide each side by 2.

Step 2

Find the \(y\)-intercept \(b\) by using \(m = -\frac{1}{2}\) and \((x, y) = (-2, 0)\).

\[
y = mx + b
\]

Use the slope-intercept form.

\[
0 = -\frac{1}{2}(-2) + b
\]

Substitute for \(m\), \(x\), and \(y\).

\[
-1 = b
\]

Solve for \(b\).

Because \(m = -\frac{1}{2}\) and \(b = -1\), an equation of the line is \(y = -\frac{1}{2}x - 1\).

Write an equation of the line passing through point \(P\) that is perpendicular to the given line.

1. \(P(3, 1), y = \frac{1}{3}x - 5\)
2. \(P(4, -3), y = -x - 5\)
3. \(P(-1, -2), y = -4x + 13\)

Writing Compound Inequalities (Math I)

Example 2

Write each sentence as an inequality.

a. A number \(x\) is greater than or equal to \(-1\) and less than 6.

\[
x \geq -1 \quad \text{and} \quad x < 6
\]

An inequality is \(-1 \leq x < 6\).

b. A number \(y\) is at most 4 or at least 9.

\[
y \leq 4 \quad \text{or} \quad y \geq 9
\]

An inequality is \(y \leq 4 \text{ or } y \geq 9\).

Write the sentence as an inequality.

4. A number \(w\) is at least \(-3\) and no more than 8.
5. A number \(m\) is more than 0 and less than 11.
6. A number \(s\) is less than or equal to 5 or greater than 2.
7. A number \(d\) is fewer than 12 or no less than \(-7\).
8. ABSTRACT REASONING: Is it possible for the solution of a compound inequality to be all real numbers? Explain your reasoning.

Dynamic Solutions available at BigIdeasMath.com

Have students make a Summary Triangle to explain how to write an equation of a perpendicular line. Include the following terms.

- Slope
- Slope-intercept form
- \(y\)-intercept

Copyright © Big Ideas Learning, LLC. All rights reserved.
MONITORING PROGRESS

ANSWERS
1. A perpendicular bisector is perpendicular to a side of the triangle at its midpoint.
2. An angle bisector divides an angle of the triangle into two congruent adjacent angles.
3. A median of a triangle is a segment from a vertex to the midpoint of the opposite side.
4. An altitude of a triangle is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side.
5. A midsegment of a triangle is a segment that connects the midpoints of two sides of the triangle.

Mathematical Practices

Lines, Rays, and Segments in Triangles

Core Concept

Lines, Rays, and Segments in Triangles

Core Concept

Perpendicular Bisector
Angle Bisector
Median

Altitude
Midsegment

EXAMPLE 1 Drawing a Perpendicular Bisector

Use dynamic geometry software to construct the perpendicular bisector of one of the sides of the triangle with vertices \( A(-1, 2), \ B(5, 4), \) and \( C(4, -1) \). Find the lengths of the two segments of the bisected side.

SOLUTION

The two segments of the bisected side have the same length, \( AD = CD = 2.92 \) units.

Monitoring Progress:

Refer to the figures at the top of the page to describe each type of line, ray, or segment in a triangle.

1. perpendicular bisector
2. angle bisector
3. median
4. altitude
5. midsegment

Laurie’s Notes

• There are many definitions that students will encounter in their study of geometry. Exploring with technology can help students discover the meaning of a word.
• Give time for students to work through the questions in the Monitoring Progress, and then discuss as a class.
Overview of Section 6.4

Introduction
• In this lesson, the medians of a triangle are constructed and the centroid, the point of concurrency, is found. The centroid is the point that is \( \frac{2}{3} \) the distance from the vertex to the midpoint of the opposite side.
• In the second part of the lesson, the three altitudes of a triangle are constructed and the orthocenter, the point of concurrency, is found.
• The lesson also integrates algebraic skills as students find the coordinates of the centroid and orthocenter.

Resources
• Tracing paper is helpful when the explorations are done by paper folding.

Teaching Strategy
• Like the last lesson, the points of concurrency in this lesson are quickly constructed using dynamic geometry software as shown in the explorations.
• If you do not have access to software, an alternate way to introduce this lesson would be through a paper-folding activity or a compass construction activity. It is powerful for students to explore and discover the properties of medians and altitudes.
• Give directions similar to what is written in the explorations for students to paper fold (or construct) the medians (Exploration 1) and the altitudes (Exploration 2). Distances can be measured using a compass or ruler. Although precision may be a problem, the focus is on the discovery.

Extensions
• There is a famous construction called the Nine-Point Circle for which you can find references in many places. When your students have finished this lesson, they will have the vocabulary necessary to perform this construction. It can be done with a compass and a straightedge or with dynamic geometry software.

Pacing Suggestion
• The explorations allow students to explore and discover two relationships that will be stated formally in the lesson. The constructions and investigations are not long and should be followed with the statement of the Centroid Theorem in the lesson. You may choose to omit the compass construction in the lesson.
Exploration

Motivate
- Before students arrive, cut a triangle out of heavier weight paper so that it remains rigid when held. You will also need a sharp pencil.
- When students arrive, ask them whether they think you could balance the triangle on the tip of your pencil.
- Explain to students that in this lesson they will learn about a special point in a triangle that is called the centroid, which is the balancing point of the triangle. (See Exercise 50.)

Exploration 1
- This is a quick and straightforward construction using dynamic geometric software. Students should be comfortable at this point knowing that the triangle will be manipulated so that various cases can be explored.
- Students observe that the three medians appear to intersect at one point.
- Turn and Talk: "How could you show that the segments intersect at just one point and not at points that are simply very close together?" One way is to label the points of intersection of each pair of segments G, H, and I. If coordinates appear, students will note that G, H, and I are the same point. If the construction was not done in a coordinate plane, measure —GH, —HI, and —IG. All segments will have a length of 0, meaning G, H, and I are the same point.

Construct Viable Arguments and Critique the Reasoning of Others: Measure the segments identified and compute the ratios. "What conjecture(s) can you make?"
Sample answer: The ratio of the length of the longer segment to the length of the median is 2:3.

- Extension: If time permits, have students construct △DEF, where D, E, and F are the midpoints of the three sides of △ABC. Ask students to make observations about the central triangle, all four small triangles, and the original △ABC.

Exploration 2
- When constructing the perpendicular line from a vertex to the opposite side of the triangle, a line will be drawn, not a segment. You may want to show students how to construct the perpendicular, draw the segment, and then hide the line.
- Construct Viable Arguments and Critique the Reasoning of Others: Students will observe that the lines containing the three altitudes are concurrent. They should also notice when they click and drag on the vertices that the point of concurrency is not always in the interior of the triangle.

Communicate Your Answer
- Students should state that the three medians of a triangle intersect at one point, and the lines containing the three altitudes of a triangle intersect at one point.

Connecting to Next Step
- The explorations are related to the theorem and properties presented in the lesson, helping students make sense of these before proving them to be true.
6.4 Medians and Altitudes of Triangles

**Essential Question** What conjectures can you make about the medians and altitudes of a triangle?

**Exploration 1** Finding Properties of the Medians of a Triangle

*a.* Plot the midpoint of $BC$ and label it $D$. Draw $AD$, which is a median of $\triangle ABC$. Construct the medians to the other two sides of $\triangle ABC$.

*b.* What do you notice about the medians? Drag the vertices to change $\triangle ABC$. Use your observations to write a conjecture about the medians of a triangle.

*c.* In the figure above, point $G$ divides each median into a shorter segment and a longer segment. Find the ratio of the length of each longer segment to the length of the whole median. Is this ratio always the same? Justify your answer.

**Exploration 2** Finding Properties of the Altitudes of a Triangle

*a.* Construct the perpendicular segment from vertex $A$ to $BC$. Label the endpoint $D$. $AD$ is an altitude of $\triangle ABC$.

*b.* Construct the altitudes to the other two sides of $\triangle ABC$. What do you notice?

*c.* Write a conjecture about the altitudes of a triangle. Test your conjecture by dragging the vertices to change $\triangle ABC$.

**Communicate your Answer**

3. What conjectures can you make about the medians and altitudes of a triangle?

4. The length of median $RU$ in $\triangle RST$ is 3 inches. The point of concurrency of the three medians of $\triangle RST$ divides $RU$ into two segments. What are the lengths of these two segments?

They meet at the same point.

*b.* The medians of a triangle are concurrent at a point inside the triangle.

*c.* 2 : 3; yes; The ratio is the same for each median and does not change when you change the triangle.

2. a. Check students’ work.

*b.* Sample answer:

They meet at the same point.

3. The medians meet at a point inside the triangle that divides each median into two segments whose lengths have the ratio 1 : 2. The altitudes meet at a point inside, on, or outside the triangle.

4. 1 in. and 2 in.
What You Will Learn
- Use medians and find the centroids of triangles.
- Use altitudes and find the orthocenters of triangles.

Using the Median of a Triangle
A **median of a triangle** is a segment from a vertex to the midpoint of the opposite side. The three medians of a triangle are concurrent. The point of concurrency, called the **centroid**, is inside the triangle.

### Centroid Theorem
The centroid of a triangle is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The medians of \( \triangle ABC \) meet at point \( P \), and 
- \( AP = \frac{2}{3}AE \)
- \( BP = \frac{2}{3}BF \)
- \( CP = \frac{2}{3}CD \)

**Proof** [BigIdeasMath.com](http://www.bigideasmath.com)

### Finding the Centroid of a Triangle
**Construction**
Use a compass and straightedge to construct the medians of \( \triangle ABC \).

**Step 1**
1. Find midpoints
   - Draw \( \triangle ABC \). Find the midpoints of \( AB \), \( BC \), and \( AC \).
   - Label the midpoints of the sides \( D \), \( E \), and \( F \), respectively.

**Step 2**
2. Draw medians
   - Draw \( AE \), \( BF \), and \( CD \). These are the three medians of \( \triangle ABC \).

**Step 3**
3. Label a point
   - Label the point where \( AE \), \( BF \), and \( CD \) intersect as \( P \). This is the centroid.

### Extra Example 1
In \( \triangle RST \), point \( Q \) is the centroid, and \( VQ = 5 \). Find \( RQ \) and \( RV \).

**SOLUTION**
- \( RQ = 10 \), \( RV = 15 \)

### Teacher Actions
- **Kinesthetic**
The centroid of a triangle is its center of gravity. Distribute cardboard to the class. Have each student draw a large triangle, construct its centroid, cut out the triangle, and try to balance it on a pencil point. The tip of the pencil will coincide with the centroid of the triangle.

- **Laurie’s Notes**
  - The Centroid Theorem follows from the first exploration. The centroid divides the median into two pieces that are in the ratio of 1:2. This also means that the ratio of the longer segment to the median is 2:3.
  - **Extension:** You can find lots of information about centroids, balancing points, and centers of gravity. The U.S. Census Bureau also defines a population centroid. Search the Internet for “population centroid.”
Finding the Centroid of a Triangle

Find the coordinates of the centroid of \( \triangle RST \) with vertices \( R(2, 1), S(5, 8), \) and \( T(8, 3) \).

**SOLUTION**

**Step 1**
Graph \( \triangle RST \).

**Step 2**
Use the Midpoint Formula to find the midpoint \( V \) of \( RT \) and sketch median \( SV \).

\[
V = \left( \frac{2 + 8}{2}, \frac{1 + 3}{2} \right) = (5, 2)
\]

**Step 3**
Find the centroid. It is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The distance from vertex \( S(5, 8) \) to \( V(5, 2) \) is \( 8 - 2 = 6 \) units.

So, the centroid is \( \frac{2}{3}(6) = 4 \) units down from vertex \( S \) on \( SV \).

So, the coordinates of the centroid \( P \) are \((5, 8 - 4)\), or \((5, 4)\).

**Monitoring Progress**

There are three paths through a triangular park. Each path goes from the midpoint of one edge to the opposite corner. The paths meet at point \( P \).

1. Find \( PS \) and \( PC \) when \( SC = 2100 \) feet.
2. Find \( TC \) and \( BC \) when \( BT = 1000 \) feet.
3. Find \( PA \) and \( TA \) when \( PT = 800 \) feet.

Find the coordinates of the centroid of the triangle with the given vertices.

4. \( F(2, 5), G(4, 9), H(6, 1) \)
5. \( X(-3, 3), Y(1, 5), Z(-1, -2) \)

**Using the Altitude of a Triangle**

An **altitude of a triangle** is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side.

**Orthocenter**

The lines containing the altitudes of a triangle are concurrent. This point of concurrency is the **orthocenter** of the triangle.

The lines containing \( AF \), \( BD \), and \( CE \) meet at the orthocenter \( G \) of \( \triangle ABC \).

---

**Laurie’s Notes**

- You might consider having different groups of students use different medians in Example 2 when finding the coordinates of the centroid.
- **Think-Pair-Share:** Have students answer Questions 1–5, and then share and discuss as a class.
- **Teaching Tip:** To sketch the three altitudes, students find it helpful to rotate their paper so that they are drawing a vertical line from the vertex, perpendicular to the opposite side. They rotate the triangle for each altitude sketched.
Extra Example 3
Find the coordinates of the orthocenter of \( \triangle DEF \) with vertices \( D(0, 6) \), \( E(-4, -2) \), and \( F(4, 6) \). \((-4, 10)\)

MONITORING PROGRESS ANSWERS
6. outside; \((-1, -3)\)
7. on; \((-3, 4)\)

As shown below, the location of the orthocenter \( P \) of a triangle depends on the type of triangle.

**EXAMPLE 3** Finding the Orthocenter of a Triangle

Find the coordinates of the orthocenter of \( \triangle XYZ \) with vertices \( X(-5, -1) \), \( Y(-2, 4) \), and \( Z(3, -1) \).

**SOLUTION**

**Step 1** Graph \( \triangle XYZ \).

**Step 2** Find an equation of the line that contains the altitude from \( Z \) to \( XZ \). Because \( XZ \) is horizontal, the altitude is vertical. The line that contains the altitude passes through \( Y(-2, 4) \). So, the equation of the line is \( x = -2 \).

**Step 3** Find an equation of the line that contains the altitude from \( X \) to \( YZ \).

\[
slope of \, YZ = \frac{-1 - 4}{5 - (-2)} = \frac{-1}{\frac{7}{3}} = -\frac{3}{7}
\]

Because the product of the slopes of two perpendicular lines is \(-1\), the slope of a line perpendicular to \( YZ \) is \( \frac{7}{3} \). The line passes through \( X(-5, -1) \).

\[
y = mx + b \quad \text{Use slope-intercept form.}
\]
\[
-1 = \frac{7}{3}(-5) + b \quad \text{Substitute } -1 \text{ for } y, \frac{7}{3} \text{ for } m, \text{ and } -5 \text{ for } x.
\]
\[
b = \frac{4}{3} \quad \text{Solve for } b.
\]

So, the equation of the line is \( y = \frac{7}{3}x + \frac{4}{3} \).

**Step 4** Find the point of intersection of the graphs of the equations \( x = -2 \) and \( y = \frac{7}{3}x + \frac{4}{3} \).

Substitute \(-2\) for \( x \) in the equation \( y = \frac{7}{3}x + \frac{4}{3} \). Then solve for \( y \).

\[
y = \frac{7}{3}(-2) + \frac{4}{3} \quad \text{Write equation.}
\]
\[
y = -\frac{14}{3} + \frac{4}{3} \quad \text{Substitute } -2 \text{ for } x.
\]
\[
y = -\frac{10}{3} \quad \text{Solve for } y.
\]

\( \triangle XYZ \) is inside triangle.

**Monitoring Progress** Tell whether the orthocenter of the triangle with the given vertices is inside, on, or outside the triangle. Then find the coordinates of the orthocenter.

6. \( A(0, 3) \), \( B(0, -2) \), \( C(6, -3) \)
7. \( J(-3, -4) \), \( K(-3, 4) \), \( L(5, 4) \)

Laurie’s Notes

**Teacher Actions**

\( \text{Is the orthocenter always located in the interior of the triangle?} \)

\( \text{No; It is in the interior for an acute triangle, at the vertex of the right angle for a right triangle, and in the exterior for an obtuse triangle.} \)

**Thumbs Up:** You might consider having different groups of students use different pairs of altitudes in Example 3 when finding the coordinates of the orthocenter. Ask students to give a Thumbs Up assessment of the process before they begin.
In an isosceles triangle, the perpendicular bisector, angle bisector, median, and altitude from the vertex angle to the base are all the same segment. In an equilateral triangle, this is true for any vertex.

**Example 4**  Proving a Property of Isosceles Triangles

Prove that the median from the vertex angle to the base of an isosceles triangle is an altitude.

**Solution**

**Given** \( \triangle ABC \) is isosceles, with base \( \overline{AC} \).

**Prove** \( BD \) is an altitude of \( \triangle ABC \).

**Paragraph Proof**

Legs \( AB \) and \( BC \) of isosceles \( \triangle ABC \) are congruent. \( CD \parallel AD \) because \( BD \) is the median to base \( AC \).

\( \triangle ADB \equiv \triangle CBD \) by the Reflexive Property of Congruence. So, \( \angle ADB \equiv \angle CBD \) by the SSS Congruence Theorem. \( \angle ADB \equiv \angle CBD \) because corresponding parts of congruent triangles are congruent. Also, \( \angle ADB \) and \( \angle CBD \) are a linear pair. \( BD \) and \( AC \) intersect to form a linear pair of congruent angles.

Thus, \( BD \parallel AC \) and \( BD \) and \( AC \) are an altitude of \( \triangle ABC \).

**Monitoring Progress**

8. **What If?** In Example 4, you want to show that median \( BD \) is also an angle bisector. How would your proof be different?

---

**Concept Summary**

<table>
<thead>
<tr>
<th>Segments, Lines, Rays, and Points in Triangles</th>
<th>Example</th>
<th>Point of Concurrency</th>
<th>Property</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>perpendicular bisector</td>
<td></td>
<td>circumcenter</td>
<td>The circumcenter ( P ) of a triangle is equidistant from the vertices of the triangle.</td>
<td></td>
</tr>
<tr>
<td>angle bisector</td>
<td></td>
<td>incenter</td>
<td>The incenter ( I ) of a triangle is equidistant from the sides of the triangle.</td>
<td></td>
</tr>
<tr>
<td>median</td>
<td></td>
<td>centroid</td>
<td>The centroid ( R ) of a triangle is two thirds of the distance from each vertex to the midpoint of the opposite side.</td>
<td></td>
</tr>
<tr>
<td>altitude</td>
<td></td>
<td>orthocenter</td>
<td>The lines containing the altitudes of a triangle are concurrent at the orthocenter ( O ).</td>
<td></td>
</tr>
</tbody>
</table>

---

**Laurie’s Notes**

**Teacher Actions**

- **Turn and Talk:** “Are the four points of concurrence distinct points in every triangle?” Listen for valid reasoning.
- **Whiteboarding:** Pose Example 4. Use whiteboards and have partners work on the proof. Compare and contrast proofs from different pairs of students.
- **Students should find the Concept Summary helpful.**

**Closure**

- **Exit Ticket:** Draw a right scalene triangle. Sketch the three medians. Draw an obtuse isosceles triangle. Sketch the three altitudes. **Check students’ work.**
ANSWERS
1. circumcenter, incenter, centroid, orthocenter; perpendicular bisectors form the circumcenter, angle bisectors form the incenter, medians form the centroid, altitudes form the orthocenter
2. 3. 6, 3
4. 14, 7
5. 20, 10
6. 7. 10, 15
8. 9. 18, 27
10. 11. 12
12. 9
13. 10
14. 5
15. (5, 11/3)
16. (−7/3, −5)
17. (5, 1)
18. (10/3, 3)
19. outside; (0, −5)
20. on; (−3, 2)
21. inside; (−1, 2)
22. inside; (0, 7/3)
23. 
24. Sample answer:
25. Sample answer:
26. Sample answer:
39. **MODELING WITH MATHEMATICS** Find the area of the triangular part of the paper airplane wing that is outlined in red. Which special segment of the triangle did you use?

40. **ANALYZING RELATIONSHIPS** Copy and complete the statement for \( \triangle DEF \) with centroid \( K \) and medians \( DH, EJ, \) and \( FG. \)
   
   a. \( EJ = \_\_\_\_\_\_\_\_\_ KJ \)
   
   b. \( DK = \_\_\_\_\_\_\_\_\_ KH \)
   
   c. \( FG = \_\_\_\_\_\_\_\_\_KF \)
   
   d. \( KG = \_\_\_\_\_\_\_\_\_FG \)

**MATHEMATICAL CONNECTIONS** In Exercises 41–44, point \( D \) is the centroid of \( \triangle ABC. \) Use the given information to find the value of \( x. \)

41. \( BD = 4x + 5 \) and \( BF = 9x \)

42. \( GD = 2x - 8 \) and \( GC = 3x + 3 \)

43. \( AD = 5x \) and \( DE = 3x - 2 \)

44. \( DF = 4x - 1 \) and \( BD = 6x + 4 \)

45. **MATHEMATICAL CONNECTIONS** Graph the lines on the same coordinate plane. Find the centroid of the triangle formed by their intersections.

   \( y_1 = 3x - 4 \)

   \( y_2 = \frac{3}{4}x + 5 \)

   \( y_3 = -\frac{3}{4}x - 4 \)

46. **CRITICAL THINKING** In what type(s) of triangles can a vertex be one of the points of concurrency of the triangle? Explain your reasoning.

### Answers

27. The length of \( DE \) should be \( \frac{1}{2} \) of the length of \( AE \) because it is the shorter segment from the centroid to the side; \( DE = \frac{1}{2}AE \)

28. The length of \( DE \) is \( \frac{1}{2} \) of the length of \( AD \); \( AD = \frac{2}{3}AE \); \( DE = \frac{1}{2}AD \); \( DE = \frac{3}{2}(24) \)

29. The orthocenter is \( \_\_\_\_\_\_\_\_\_ \) outside the triangle.

30. The altitude from the vertex angle to the base of an isosceles triangle is also a perpendicular bisector.

31. **CRITICAL THINKING** In Exercises 31–36, complete the statement with always, sometimes, or never. Explain your reasoning.

   a. The centroid is \( \_\_\_\_\_\_\_\_\_ \) on the triangle.

   b. The orthocenter is \( \_\_\_\_\_\_\_\_\_ \) outside the triangle.

   c. A median is \( \_\_\_\_\_\_\_\_\_ \) the same line segment as a perpendicular bisector.

   d. An altitude is \( \_\_\_\_\_\_\_\_\_ \) the same line segment as an angle bisector.

   e. The centroid and orthocenter are \( \_\_\_\_\_\_\_\_\_ \) the same point.

   f. The centroid is \( \_\_\_\_\_\_\_\_\_ \) formed by the intersection of the three medians.

32. A median is \( \_\_\_\_\_\_\_\_\_ \) the same line segment as a perpendicular bisector.

33. An altitude is \( \_\_\_\_\_\_\_\_\_ \) the same line segment as an angle bisector.

34. The centroid and orthocenter are \( \_\_\_\_\_\_\_\_\_ \) the same point.

35. The centroid and orthocenter are \( \_\_\_\_\_\_\_\_\_ \) the same segment.

36. Sometimes: An altitude is the same line segment as the angle bisector if the triangle is equilateral or if the segment is connecting the vertex angle to the base of an isosceles triangle. Otherwise, the altitude and the angle bisector are not the same segment.

### Section 6.4 Medians and Altitudes of Triangles

31. **ERROR ANALYSIS** In Exercises 27 and 28, describe and correct the error in finding \( DE. \) Point \( D \) is the centroid of \( \triangle ABC. \)

32. \( DE = \frac{2}{3} \) \(AE\)

33. \( DE = \frac{2}{3} \) \(AE\)

34. \( DE = \frac{2}{3} \) \(AE\)

35. \( DE = \frac{2}{3} \) \(AE\)

36. \( DE = \frac{2}{3} \) \(AE\)

37. \( DE = \frac{2}{3} \) \(AE\)

38. \( DE = \frac{2}{3} \) \(AE\)

39. \( DE = \frac{2}{3} \) \(AE\)

40. \( DE = \frac{2}{3} \) \(AE\)

41. \( DE = \frac{2}{3} \) \(AE\)

42. \( DE = \frac{2}{3} \) \(AE\)

43. \( DE = \frac{2}{3} \) \(AE\)

44. \( DE = \frac{2}{3} \) \(AE\)

45. \( DE = \frac{2}{3} \) \(AE\)

46. \( DE = \frac{2}{3} \) \(AE\)

### Additional Answers

31. never; Because medians are always inside a triangle, and the centroid is the point of concurrency of the medians, it will always be inside the triangle.

32. sometimes; An orthocenter can be inside, on, or outside the triangle depending on whether the triangle is acute, right, or obtuse.

33. sometimes; A median is the same line segment as the perpendicular bisector if the triangle is equilateral or if the segment is connecting the vertex angle to the base of an isosceles triangle. Otherwise, the median and the perpendicular bisectors are not the same segment.

34. sometimes; An altitude is the same line segment as the angle bisector if the triangle is equilateral or if the segment is connecting the vertex angle to the base of an isosceles triangle. Otherwise, the altitude and the angle bisector are not the same segment.

35. sometimes; The centroid and orthocenter are not the same point unless the triangle is equilateral.

36–46. See Additional Answers.
47. **WRITING EQUATIONS** Use the numbers and symbols to write three different equations for $PE$. 

48. **HOW DO YOU SEE IT?** Use the figure.

49. **MAKING AN ARGUMENT** Your friend claims that it is possible for the circumcenter, incenter, centroid, and orthocenter to all be the same point. Do you agree? Explain your reasoning.

50. **DRAWING CONCLUSIONS** The center of gravity of a triangle, the point where a triangle can balance on the tip of a pencil, is one of the four points of concurrency. Draw and cut out a large scalene triangle on a piece of cardboard. Which of the four points of concurrency is the center of gravity? Explain.

51. **PROOF** Prove that a median of an equilateral triangle is also an angle bisector, perpendicular bisector, and altitude.

52. **THOUGHT PROVOKING** Construct an acute scalene triangle. Find the orthocenter, centroid, and circumcenter. What can you conclude about the three points of concurrency?

53. **CONSTRUCTION** Follow the steps to construct a nine-point circle. Why is it called a nine-point circle?

54. **PROOF** Prove the statements in parts (a)–(c).

55. **MAINTAINING MATHEMATICAL PROFICIENCY** Reviewing what you learned in previous grades and lessons.

56. **Mini-Assessment**

1. Find the coordinates of the centroid of $\triangle ABC$ with vertices $A(−1, 5)$, $B(−3, −2)$, and $C(1, 0)$. ($−1, 1$)

2. Point $P$ is the centroid of $\triangle LMN$, and $QN = 12.6$. Find $PN$ and $QP$.

3. Find the coordinates of the orthocenter of $\triangle DEF$ with vertices $D(−1, −3)$, $E(−1, 5)$, and $F(3, −1)$. ($1, −1$)
6.1–6.4 What Did You Learn?

Core Vocabulary

- proof, p. 336
- two-column proof, p. 336
- paragraph proof, p. 337
- flowchart proof, or flow proof, p. 338
- coordinate proof, p. 339
- equidistant, p. 344
- concurrent, p. 352
- point of concurrency, p. 352
- circumcenter, p. 352
- incenter, p. 355
- median of a triangle, p. 362
- centroid, p. 362
- altitude of a triangle, p. 363
- orthocenter, p. 363

Core Concepts

Section 6.1
- Writing Two-Column Proofs, p. 336
- Writing Paragraph Proofs, p. 337
- Writing Flowchart Proofs, p. 338
- Writing Coordinate Proofs, p. 339

Section 6.2
- Perpendicular Bisector Theorem, p. 344
- Converse of the Perpendicular Bisector Theorem, p. 344
- Angle Bisector Theorem, p. 346
- Converse of the Angle Bisector Theorem, p. 346

Section 6.3
- Circumcenter Theorem, p. 352
- Incenter Theorem, p. 355

Section 6.4
- Centroid Theorem, p. 362
- Segments, Lines, Rays, and Points in Triangles, p. 365
- Orthocenter, p. 363

Mathematical Practices

1. Did you make a plan before completing your proof in Exercise 37 on page 350? Describe your thought process.
2. What tools did you use to complete Exercises 17–20 on page 358? Describe how you could use technological tools to complete these exercises.

Rework Your Notes

A good way to reinforce concepts and put them into your long-term memory is to rework your notes. When you take notes, leave extra space on the pages. You can go back after class and fill in

- important definitions and rules,
- additional examples, and
- questions you have about the material.

ANSWERS

1. Sample answer: yes; You realize that if you construct the segment $AC$, you have created two isosceles triangles that share vertices $A$ and $B$, and a third triangle that shares the same vertices. Then, you look back at the Perpendicular Bisector Theorem and its converse to see that points $D$ and $E$ would have to be on the perpendicular bisector of $AC$. Then, in the same way, in order for $B$ to also be on the same perpendicular bisector, $AB$ and $CB$ would have to be congruent.

2. Sample answer: These can be constructed with a compass and a straightedge, or they can be constructed with geometry software; If you construct them with geometry software, you create a triangle that fits the description first. Then, use the software to draw the perpendicular bisectors of each side. Next, label the point where these three lines meet. Finally, draw a circle with its center at this point of intersection that passes through one vertex of the triangle. It will automatically pass through the other two vertices.
ANSWERS
1. 15; Because \( SW \parallel UW \) and \( VW \perp SU \), point \( V \) is on the perpendicular bisector of \( SU \). So, by the Perpendicular Bisector Theorem, \( SV = UV \). So, \( 2x + 11 = 8x - 1 \), and the solution is \( x = 2 \), which means that \( UV = 8x - 1 = 8(2) - 1 = 15 \).

2. 18; \( SQ \) is an angle bisector of \( \angle PSR \), \( PQ \perp SP \), and \( RQ \perp SR \). So, by the Angle Bisector Theorem, \( PQ = RQ \). So, \( 6x = 3x + 9 \), and the solution is \( x = 3 \), which means that \( QP = 6x = 6(3) = 18 \).

3. 59°; Because \( J \) is equidistant from \( GH \) and \( GK \), \( GJ \) bisects \( \angle HGK \) by the Angle Bisector Theorem. This means that \( 5x - 4 = 4x + 3 \), and the solution is \( x = 7 \). So, \( m\angle JGK = 4x + 3 = 4(7) + 3 = 31° \). So, by the Triangle Sum Theorem, \( m\angle GJK = 180° - 90° - 31° = 59° \).

4. \((-2, -1)\)
5. \((7, 5)\)
6. 11
7. 15
8. 26
9. \((3, 2)\)
10. \((-4, -4)\)
11. inside; \((0, 4)\)
12. on; \((7, -4)\)

13. a. incenter; angle bisectors
b. From the diagram, \( EG \perp AB \) and \( FG \perp CB \). By the definition of perpendicular lines, \( \angle BEG \) and \( \angle BFG \) are right angles. So, \( \triangle BEG \) and \( \triangle BFG \) are right triangles, by the definition of a right triangle. From part (a), \( G \) is the incenter of \( \triangle ABC \). So, by the Incenter Theorem, \( GE = GF = GD \). Because \( GE = GF, GE \cong GF \), by the definition of congruent segments. By the Reflexive Property of Congruence, \( BG \cong BG \). So, by the Hypotenuse-Leg Congruence theorem, \( \triangle BGF \cong \triangle BGE \).
c. 3.9 cm; Because \( \triangle BGF \cong \triangle BGE \) and corresponding parts of congruent triangles are congruent, \( BE = BF = 3 \) centimeters. So, \( AE = 10 - 3 = 7 \) centimeters. Then, you can use the Pythagorean Theorem for \( \triangle AEG \) to find \( EG \), which is the radius of the wheel.

14. a. centroid
b. no; This point is not equidistant from the three cities. The circumcenter would be equidistant from the cities.

370 Chapter 6 Relationships Within Triangles
6.5–6.7 What Did You Learn?

Core Vocabulary

- midsegment of a triangle, p. 372
- indirect proof, p. 378

Core Concepts

**Section 6.5**

Using the Midsegment of a Triangle, p. 372
Triangle Midsegment Theorem, p. 373

**Section 6.6**

How to Write an Indirect Proof (Proof by Contradiction), p. 378
Triangle Longer Side Theorem, p. 379
Triangle Larger Angle Theorem, p. 379
Triangle Inequality Theorem, p. 381

**Section 6.7**

Hinge Theorem, p. 386
Converse of the Hinge Theorem, p. 386

Mathematical Practices

1. In Exercise 25 on page 376, analyze the relationship between the stage and the total perimeter of all the shaded triangles at that stage. Then predict the total perimeter of all the shaded triangles in Stage 4.

2. In Exercise 17 on page 382, write all three inequalities using the Triangle Inequality Theorem. Determine the reasonableness of each one. Why do you only need to use two of the three inequalities?

3. In Exercise 23 on page 390, try all three cases of triangles (acute, right, obtuse) to gain insight into the solution.

Performance Task:

Building a Roof Truss

The simple roof truss is also called a planar truss because all its components lie in a two-dimensional plane. How can this structure be extended to three-dimensional space? What applications would this type of structure be used for?

To explore the answers to these questions and more, check out the Performance Task and Real-Life STEM video at BigIdeasMath.com.

ANSWERS

1. Let \( n \) be the stage, then the side length of the new triangles in each stage is \( 2^{4-n} \). So, the perimeter of each new triangle is \( (3 \cdot 2^{4-n}) \). The number of new triangles is given by \( (3^n - 1) \). So, to find the perimeter of all the shaded triangles in each stage, start with the total from the previous stage and add \( (3 \cdot 2^{4-n})(3^n - 1) \).

   The perimeter of the new triangles in stage 4 will be \( (3 \cdot 2^{4-4})(3^4 - 1) \)
   \( = 81 \). The total perimeter of the new triangles and old triangles is \( 81 + 114 = 195 \) units.

2. \( x + 5 > 12, x + 12 > 5, 5 + 12 > x; \) Because the length of the third side has to be a positive value, the inequality \( x + 12 > 5 \) will always be true. So, you do not have to consider this inequality in determining the possible values of \( x \). Solve the other two inequalities to find that the length of the third side must be greater than 7 and less than 19.

3. If \( m\angle BAC < m\angle BDC \), then the orthocenter \( D \) is inside the triangle, and \( \triangle ABC \) is an acute triangle. If \( m\angle BAC = m\angle BDC \), then the orthocenter \( D \) is on vertex \( A \), and \( \triangle ABC \) is a right triangle, with \( \angle A \) being the right angle. If \( m\angle BAC > m\angle BDC \), then the orthocenter \( D \) is outside the triangle, and \( \triangle ABC \) is an obtuse triangle, with \( \angle A \) being the obtuse angle.
ANSWERS

1. \( m\angle 5 + m\angle 7 = 90^\circ \). \( \angle 5 \) and \( \angle 6 \) are complementary. By the definition of complementary angles, 
   \( m\angle 5 + m\angle 6 = 90^\circ \). So, by the 
   Transitive Property of Equality, 
   \( m\angle 5 + m\angle 6 = m\angle 5 + m\angle 7 \). 
   \( m\angle 6 = m\angle 7 \) by the Subtraction 
   Property of Equality. So, by the 
   definition of congruent angles, 
   \( \angle 6 \equiv \angle 7 \).

2. By the Distance Formula, 
   \( OB = \sqrt{a^2 + b^2} \) and \( OC = \sqrt{a^2 + b^2} \). 
   So, by the definition of congruent 
   segments, \( OB \equiv OC \) and \( \triangle OBC \) is 
   isosceles by definition.

### 6.1 Proving Geometric Relationships (pp. 335–342)

#### a. Rewrite the two-column proof into a paragraph proof.

**Given** 
\( \angle 6 \equiv \angle 7 \)

**Prove** 
\( \angle 7 \equiv \angle 2 \)

**Two-Column Proof**

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle 6 \equiv \angle 7 )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \angle 6 \equiv \angle 2 )</td>
<td>2. Vertical Angles Congruence Theorem</td>
</tr>
<tr>
<td>3. ( \angle 7 \equiv \angle 2 )</td>
<td>3. Transitive Property of Congruence</td>
</tr>
</tbody>
</table>

**Paragraph Proof**

\( \angle 6 \) and \( \angle 7 \) are congruent. By the Vertical Angles Congruence Theorem, \( \angle 6 \equiv \angle 2 \). 
So, by the Transitive Property of Congruence, \( \angle 7 \equiv \angle 2 \).

#### b. Write a coordinate proof.

**Given** 
Coordinates of vertices of \( \triangle OAC \) and \( \triangle BAC \)

**Prove** 
\( \triangle OAC \equiv \triangle BAC \)

**Segments** \( OA \) and \( BA \) have the same length, so \( OA \equiv BA \).

\( OA = \sqrt{c^2 - 0} = \frac{c}{2} \)

\( BA = \sqrt{c^2 - c^2} = \frac{c}{2} \)

**Segments** \( OC \) and \( BC \) have the same length, so \( OC \equiv BC \).

\( OC = \sqrt{(0 - c)^2 + (0 - c)^2} = \sqrt{-\frac{c^2}{2} + (-c)^2} = \sqrt{\frac{c^2}{4}} = \frac{c}{2} \sqrt{5} \)

\( BC = \sqrt{(c - c)^2 + (0 - c)^2} = \sqrt{0 + (-c)^2} = \sqrt{\frac{c^2}{4}} = \frac{c}{2} \sqrt{5} \)

By the Reflexive Property of Congruence, \( AC \equiv AC \).

So, you can apply the SSS Congruence Theorem to conclude that \( \triangle OAC \equiv \triangle BAC \).

1. Use the figure in part (a). Write a proof using any format.

**Given** 
\( \angle 5 \) and \( \angle 6 \) are complementary. 
\( m\angle 5 + m\angle 6 = 90^\circ \)

**Prove** 
\( \angle 6 \equiv \angle 7 \)

2. Use the figure at the right. Write a coordinate proof.

**Given** 
Coordinates of vertices of \( \triangle OBC \)

**Prove** 
\( \triangle OBC \) is isosceles.
6.2 Perpendicular and Angle Bisectors (pp. 343–350)

Find each measure.

a. **AD**

From the figure, \( \overline{AC} \) is the perpendicular bisector of \( \overline{BD} \).

\[
\begin{align*}
AB &= AD & \text{Perpendicular Bisector Theorem} \\
4x + 3 &= 6x - 9 & \text{Substitute} \\
x &= 6 & \text{Solve for } x.
\end{align*}
\]

So, \( AD = 6(6) - 9 = 27 \).

b. **FG**

Because \( EH = GH \) and \( \overline{HF} \perp \overline{EG} \), \( \overline{HF} \) is the perpendicular bisector of \( \overline{EG} \) by the Converse of the Perpendicular Bisector Theorem. By the definition of segment bisector, \( EF = FG \).

So, \( FG = EF = 6 \).

c. **LM**

\[
\begin{align*}
\text{JM} &= \text{LM} & \text{Angle Bisector Theorem} \\
8x - 15 &= 3x & \text{Substitute} \\
x &= 3 & \text{Solve for } x.
\end{align*}
\]

So, \( LM = 3(3) = 9 \).

d. **\( m \angle XZW \)**

Because \( \overline{ZX} \perp \overline{WX} \) and \( \overline{ZY} \perp \overline{WY} \) and \( ZX = ZY = 8 \), \( WZ \) bisects \( \angle XWY \) by the Converse of the Angle Bisector Theorem.

So, \( m \angle XZW = m \angle YZW = 46^\circ \).

Find the indicated measure. Explain your reasoning.

3. **DC**

4. **RS**

5. **\( m \angle JFH \)**
6.3 Bisectors of Triangles (pp. 351–360)

a. Find the coordinates of the circumcenter of \( \triangle QRS \) with vertices \( Q(3, 3) \), \( R(5, 7) \), and \( S(9, 3) \).

Step 1 Graph \( \triangle QRS \).

Step 2 Find equations for two perpendicular bisectors.

The midpoint of \( QS \) is \((6, 3)\). The line through \((6, 3)\) that is perpendicular to \( QS \) is \( x = 6 \).

The midpoint of \( QR \) is \((4, 5)\). The line through \((4, 5)\) that is perpendicular to \( QR \) is \( y = -\frac{1}{2}x + 7 \).

Step 3 Find the point where \( x = 6 \) and \( y = -\frac{1}{2}x + 7 \) intersect. They intersect at \((6, 4)\).

So, the coordinates of the circumcenter are \((6, 4)\).

b. Point \( N \) is the incenter of \( \triangle ABC \). In the figure shown, \( ND = 4x + 7 \) and \( NE = 3x + 9 \). Find \( NF \).

Step 1 Solve for \( x \).

\[
\begin{align*}
ND &= NE \\
4x + 7 &= 3x + 9 \\
x &= 2
\end{align*}
\]

Step 2 Find \( ND \) or \( NE \).

\[
ND = 4x + 7 = 4(2) + 7 = 15
\]

So, because \( ND = NF, NF = 15 \).

Find the coordinates of the circumcenter of the triangle with the given vertices.

6. \( T(-6, -5), U(0, -1), V(0, -5) \)
7. \( X(-2, 1), Y(2, -3), Z(6, -3) \)
8. Point \( D \) is the incenter of \( \triangle LMN \). Find the value of \( x \).
6.4 Medians and Altitudes of Triangles (pp. 361–368)

Find the coordinates of the centroid of \( \triangle TUV \) with vertices \( T(1, \ -8) \), \( U(4, \ -1) \), and \( V(7, \ -6) \).

Step 1 Graph \( \triangle TUV \).

Step 2 Use the Midpoint Formula to find the midpoint \( W \) of \( TV \).

\[
W = \left( \frac{1 + 7}{2}, \frac{-8 + (-6)}{2} \right) = (4, \ -7)
\]

Step 3 Find the centroid. It is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The distance from vertex \( U(4, \ -1) \) to \( W(4, \ -7) \) is \( 1 - (-7) = 6 \) units.

So, the centroid is \( \frac{2}{3}(6) = 4 \) units down from vertex \( U \) on \( UW \).

So, the coordinates of the centroid \( P \) are \((4, \ -1 - 4)\), or \((4, \ -5)\).

Find the coordinates of the centroid of the triangle with the given vertices.

9. \( A(-10, \ 3), B(-4, \ 5), C(-4, \ 1) \)

10. \( D(2, \ -8), E(2, \ -2), F(8, \ -2) \)

Tell whether the orthocenter of the triangle with the given vertices is inside, on, or outside the triangle. Then find the coordinates of the orthocenter.

11. \( G(1, \ 6), H(5, \ 6), J(3, \ 1) \)

12. \( K(-8, \ 5), L(-6, \ 3), M(0, \ 5) \)

6.5 The Triangle Midsegment Theorem (pp. 371–376)

In \( \triangle JKL \), show that midsegment \( MN \) is parallel to \( JL \) and that \( MN = \frac{1}{2} JL \).

Step 1 Find the coordinates of \( M \) and \( N \) by finding the midpoints of \( JK \) and \( KL \).

\[
M = \left( \frac{-8 + (-4)}{2}, \frac{1 + 7}{2} \right) = M\left( \frac{-12}{2}, \frac{8}{2} \right) = M(-6, \ 4)
\]

\[
N = \left( \frac{-4 + (-2)}{2}, \frac{7 + 3}{2} \right) = N\left( \frac{-6}{2}, \frac{10}{2} \right) = N(-3, \ 5)
\]

Step 2 Find and compare the slopes of \( MN \) and \( JL \).

Slope of \( MN = \frac{5 - 4}{-3 - (-6)} = \frac{1}{3} \)

Slope of \( JL = \frac{-2 - (-8)}{6 - 3} = \frac{6}{3} \)

Because the slopes are the same, \( MN \) is parallel to \( JL \).

Step 3 Find and compare the lengths of \( MN \) and \( JL \).

\[
MN = \sqrt{(-3 - (-6))^2 + (5 - 4)^2} = \sqrt{9 + 1} = \sqrt{10}
\]

\[
JL = \sqrt{(-2 - (-8))^2 + (3 - 1)^2} = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10}
\]

Because \( \sqrt{10} = \frac{1}{2} (2\sqrt{10}) \), \( MN = \frac{1}{2} JL \).

Find the coordinates of the vertices of the midsegment triangle for the triangle with the given vertices.

13. \( A(-6, \ 8), B(-6, \ 4), C(0, \ 4) \)

14. \( D(-3, \ 1), E(3, \ 5), F(1, \ -5) \)
Given that \( \overline{WZ} = \overline{YZ}, \) how does \( \overline{XY} \) compare to \( \overline{XW} \)?

You are given that \( \overline{WZ} = \overline{YZ}, \) and you know that \( \overline{WX} = \overline{XW} \) by the Reflexive Property of Congruence.

Because \( 90° > 80°, m\angle XZY > m\angle XZW, \) so, two sides of \( \triangle XZY \) are congruent to two sides of \( \triangle XZW \) and the included angle in \( \triangle XZY \) is larger.

\[ \text{By the Hinge Theorem, } \angle XY > \angle XW. \]

Use the diagram.

19. If \( RQ = RS \) and \( m\angle QRT > m\angle SRT, \) then how does \( \overline{QT} \) compare to \( \overline{ST}? \)

20. If \( RQ = RS \) and \( \overline{QT} > \overline{ST}, \) then how does \( \angle QRT \) compare to \( \angle SRT? \)
6 Chapter Test

In Exercises 1 and 2, \( \overline{MN} \) is a midsegment of \( \triangle JKL \). Find the value of \( x \).

1. \( m = 12 \) \\
2. \( n = 17 \)

Find the indicated measure. Identify the theorem you use.

3. \( ST \) \\
4. \( WY \) \\
5. \( BW \)

Copy and complete the statement with \( < \), \( > \), or \( = \).

6. \( AB \) ___ \( CB \) \\
7. \( m \angle 1 \) ___ \( m \angle 2 \) \\
8. \( m \angle MNP \) ___ \( m \angle NPM \)

9. Find the coordinates of the circumcenter, orthocenter, and centroid of the triangle with vertices \( A(0, -2) \), \( B(4, -2) \), and \( C(0, 6) \).

10. Write an indirect proof of the Corollary to the Base Angles Theorem: If \( \triangle PQR \) is equilateral, then it is equiangular.

11. \( \triangle DEF \) is a right triangle with area \( A \). Use the area for \( \triangle DEF \) to write an expression for the area of \( \triangle GEH \). Justify your answer.

12. Prove the Corollary to the Triangle Sum Theorem using any format. Given \( \triangle ABC \) is a right triangle. Prove \( \angle A \) and \( \angle B \) are complementary.

In Exercises 13–15, use the map.

13. Describe the possible lengths of Pine Avenue.

14. You ride your bike along a trail that represents the shortest distance from the beach to Main Street. You end up exactly halfway between your house and the movie theater. How long is Pine Avenue? Explain.

15. A market is the same distance from your house, the movie theater, and the beach. Copy the map and locate the market.

Chapter 6 Chapter Test 397

If students need help...

If students got it...

Lesson Tutorials Resources by Chapter

• Enrichment and Extension

• Cumulative Review

Skills Review Handbook Performance Task

BigIdeasMath.com Start the next Section

ANSWERS

1. \( x = 6 \) \\
2. \( x = 9 \) \\
3. \( ST = 17 \); Perpendicular Bisector Theorem

4. \( WY = 32 \); Angle Bisector Theorem

5. \( BW = 20 \); Incenter Theorem

6. \( AB > CB \) \\
7. \( m \angle 1 < m \angle 2 \) \\
8. \( m \angle MNP < m \angle NPM \) \\
9. \( (2, 2); (0, -2); \left(\frac{4}{3}, \frac{1}{3}\right) \)

10. Assume temporarily that \( \triangle PQR \) is equilateral and equiangular. Then it follows that \( m \angle P \neq m \angle Q \), \( m \angle Q \neq m \angle R \), or \( m \angle P \neq m \angle R \). By the contrapositive of the Base Angles Theorem, if \( m \angle P \neq m \angle Q \), then \( PR \neq QR \), if \( m \angle Q \neq m \angle R \), then \( QP \neq RP \), and if \( m \angle P \neq m \angle R \), then \( PQ \neq RQ \). All three conclusions contradict the fact that \( \triangle PQR \) is equilateral. So, the temporary conclusion must be false. This proves that if \( \triangle PQR \) is equilateral, it must also be equiangular.

11. area of \( \triangle GEH = \frac{1}{4} A \); By the Triangle Midsegment Theorem, \( GH = \frac{1}{2} FD \). By the markings \( EG = GD \). By the Segment Addition Postulate, \( EG + GD = ED \). So, when you substitute \( EG \) for \( GD \), you get \( EG + EG = ED \), or \( 2(EG) = ED \), which means that \( EG = \frac{1}{2} ED \). So, the area of \( \triangle GEH = \frac{1}{2} bh \) \\

\[ = \frac{1}{2}(EG)(GH) \] \\
\[ = \frac{1}{2}(ED)(FD) \] \\
\[ = \frac{1}{4}(ED)(FD) \]

Note that the area of \( \triangle DEF = \frac{1}{2}bh = \frac{1}{2}(ED)(FD) \). So, the area of \( \triangle GEH = \frac{1}{4}(ED)(FD) = \frac{1}{4} \left(\frac{1}{2}(ED)(FD)\right) = \frac{1}{4} A \).

12. See Additional Answers.

13. Pine Avenue must be longer than 2 miles and shorter than 16 miles.

14–15. See Additional Answers.
1. Which definition(s) and/or theorem(s) do you need to use to prove the Converse of the Perpendicular Bisector Theorem? Select all that apply.

Given: \( CA = CB \)

Prove: Point \( C \) lies on the perpendicular bisector of \( \overline{AB} \).

- Definition of perpendicular bisector
- Definition of angle bisector
- Definition of segment congruence
- Definition of angle congruence
- Base Angles Theorem
- Converse of the Base Angles Theorem
- ASA Congruence Theorem
- AAS Congruence Theorem

2. Which of the following values are \( x \)-coordinates of the solutions of the system?

\[
\begin{align*}
y &= x^2 - 3x + 3 \\
y &= 3x - 5
\end{align*}
\]

- \(-7\), \(-4\), \(-3\), \(-2\), \(-1\)
- \(1\), \(2\), \(3\), \(4\), \(7\)

3. What are the coordinates of the centroid of \( \triangle LMN \)?

- (A) (2, 5)
- (B) (3, 5)
- (C) (4, 5)
- (D) (5, 5)

4. Use the steps in the construction to explain how you know that the circle is circumscribed about \( \triangle ABC \).

- Step 1
- Step 2
- Step 3

ANSWERS

1. definition of perpendicular bisector, definition of segment congruence

2. 2, 4

3. B

4. In step 1, the constructed line connects two points that are each equidistant from both \( A \) and \( B \). So, it is the perpendicular bisector of \( \overline{AB} \), and therefore every point on the line is equidistant from \( A \) and \( B \). In step 2, the constructed line connects two points that are each equidistant from both \( B \) and \( C \). So, it is the perpendicular bisector of \( \overline{BC} \), and therefore every point on the line is equidistant from \( B \) and \( C \). So, the point where these two lines intersect is equidistant from all three points. So, the circle with this point as the center that passes through one of the points will also pass through the other two.
5. According to a survey, 58% of voting-age citizens in a town are planning to vote in an upcoming election. You randomly select 10 citizens to survey.
   a. What is the probability that less than 4 citizens are planning to vote?
   b. What is the probability that exactly 4 citizens are planning to vote?
   c. What is the probability that more than 4 citizens are planning to vote?

6. What are the solutions of \(2x^2 + 18 = -6\)?
   (A) \(i\sqrt{6}\) and \(-i\sqrt{6}\)
   (B) \(2\sqrt{3}\) and \(-2\sqrt{3}\)
   (C) \(i\sqrt{15}\) and \(-i\sqrt{15}\)
   (D) \(-i\sqrt{21}\) and \(i\sqrt{21}\)

7. Use the graph of \(\triangle QRS\).

   a. Find the coordinates of the vertices of the midsegment triangle. Label the vertices \(T\), \(U\), and \(V\).
   b. Show that each midsegment joining the midpoints of two sides is parallel to the third side and is equal to half the length of the third side.

8. The graph of which inequality is shown?
   (A) \(y > x^2 - 3x + 4\)
   (B) \(y \geq x^2 - 3x + 4\)
   (C) \(y < x^2 - 3x + 4\)
   (D) \(y \leq x^2 - 3x + 4\)
Integrated Mathematics II Ancillaries

Assessment Book ........................................................... A2
Resources by Chapter ...................................................... A5
Student Journal ............................................................ A8
Differentiating the Lesson ............................................... A10
The Assessment Book contains formative and summative assessment options, providing teachers with the ability to assess students on the same content in a variety of ways. It is available in print and online in an editable format.

**Prerequisite Skills Test with Item Analysis**

The Prerequisite Skills Test checks students’ understanding of previously learned mathematical skills they will need to be successful in their math class. You can use the Item Analysis to diagnose topics your students need to review to prepare them for the school year.

**Pre-Course Test with Item Analysis/Post-Course Test with Item Analysis**

The Pre-Course Test and Post-Course Test cover key concepts that students will learn in their math class. You can gauge how much your students learned throughout the year by comparing their Pre-Test score against their Post-Test score. These tests also can be given as practice for state assessments.
Quiz

The Quiz provides ongoing assessment of student understanding. You can use this as a gradable quiz or as practice.

Chapter Tests

The Chapter Tests provide assessment of student understanding of key concepts taught in the chapter. There are two tests for each chapter. You can use these as gradable tests or as practice for your students to master an upcoming test.

Alternative Assessment with Scoring Rubric

Each Alternative Assessment includes at least one multi-step problem that combines a variety of concepts from the chapter. Students are asked to explain their solutions, write about the mathematics, or compare and analyze different situations. You can use this as an alternative to traditional tests. It can be graded, assigned as a project, or given as practice.
**Performance Task**

The Performance Task presents an assessment in a real-world situation. Every chapter has a task that allows students to work with multiple standards and apply their knowledge to realistic scenarios. You can use this task as an in-class project, a take-home assignment or as a graded assessment.

**Quarterly Standards Based Test**

The Quarterly Standards Based Test provides students practice answering questions in standardized test format. The assessments are cumulative and cover material from multiple chapters of the textbook. The questions represent problem types and reasoning patterns frequently found on standardized tests. You can give this test to your students as a cumulative assessment for the quarter or as practice for state assessment.
Family Communication Letters (English and Spanish)

The Family Communication Letters provide a way to quickly communicate to family members how they can help their student with the material of the chapter. You can send this home with your students to help make the mathematics less intimidating and provide suggestions for families to help their students see mathematical concepts in common activities.

Directions:

1. Plot the three points (–6, 2), (7, 8), and (10, 2), and connect the points to make a triangle.
2. Plot the two points (2, 10) and (2, −5), and draw a line through these points. Label this a perpendicular bisector.
3. Draw a point at (2, 2) where the bottom of the triangle and the perpendicular bisector intersect. Draw a line segment from (7, 8) to (2, 2). Label this a median.
4. Draw a point at (7, 2) on the bottom side of the triangle. Draw a line segment from (7, 8) to (7, 2). Label this an altitude.

Nice job! Now that you have drawn the complete picture, maybe you can do some extra analyzing on your own.

- What are two features of the perpendicular bisector in relationship to the angle it makes and the line segment it intercepts?
- What kind of angle does an altitude make with its opposite side?

Being able to see and use these concepts will not only help you along the way in your math class, it may even help you get an equal size piece of cake as those you are sharing it with!
Start Thinking/Warm Up/ Cumulative Review Warm Up

Each Start Thinking/Warm Up/ Cumulative Review Warm Up includes two options for getting the class started. The Start Thinking questions provide students with an opportunity to discuss thought-provoking questions and analyze real-world situations. You can use this to begin a class discussion. The Warm Up questions review prerequisite skills needed for the lesson. You can give this to students at the beginning of class to prepare them for the upcoming lesson. The Cumulative Review Warm Up questions review material from earlier lessons or courses. You can give these to students to continue their mastery of concepts they have been taught.

Practice

The Practice exercises provide additional practice on the key concepts taught in the lesson. There are two levels of practice provided for each lesson: A (basic) and B (average). You can assign these exercises for students that need the extra work or as a gradable assignment.

6.3 Practice A

In Exercises 1–3, the perpendicular bisectors of \(\triangle ABC\) intersect at point \(G\), or the angle bisectors of \(\triangle XYZ\) intersect at point \(P\). Find the indicated measure. Tell which theorem you used.

1. \(BG\) 2. \(CG\) 3. \(PS\)

In Exercises 4 and 5, find the coordinates of the circumcenter of the triangle with the given vertices.

4. \((0, 0), (0, 4), (3, 0)\) 5. \((0, 0), (0, 4), (3, 0)\)

In Exercises 6 and 7, \(P\) is the incenter of \(\triangle QRS\). Use the given information to find the indicated measure.

6. \(PQ = 4x - 5, PL = x + 7\) Find \(PL\)

7. \(PS + 8x - 2, PM = 8x - 14\) Find \(PL\)

8. Draw an obtuse isosceles triangle. Find the circumcenter. Then construct the circumscribed circle.

9. A cellular phone company is building a tower at an equal distance from three large apartment buildings. Explain how you can use the figure at the right to determine the location of the cell tower.

10. Your friend says that it is impossible for the circumcenter of a triangle to lie outside the triangle. Is your friend correct? Explain your reasoning.
6.3 Enrichment and Extension

Bisectors of Triangles

1. Consider the point P(3, 4). Find the point Q for which the line 2x + y = 5
   passes through the perpendicular bisector of PQ.

2. A triangle has side lengths of 14, 16, and 20 units. What is the radius of the
   circumscribed circle?

In Exercises 3 and 4, use the following information to find the coordinates of the
bisectors of the triangle.

3. Given: a = 3, b = 4, c = 5
   Find the coordinates of the circumscribed circle.

4. Given: a = 5, b = 12, c = 13
   Find the coordinates of the circumscribed circle.

5. Write an equation for the total cost of a large pizza with toppings.
   a. Toppings: cheese, pepperoni, anchovies
   b. Toppings: mushrooms, jalapenos, onions
   c. Toppings: anchovies, pepperoni, jalapenos
   d. Toppings: mushrooms, jalapenos, anchovies
   e. Toppings: anchovies, mushrooms, jalapenos
   f. Toppings: pepperoni, jalapenos, anchovies

Each Enrichment and Extension extends the lesson and provides a challenging
application of the key concepts. These rigorous exercises can be assigned to
challenge your students to use higher order thinking skills or to build on a
concept via an extension of the lesson.

Puzzle Time

What Did The Computer Do At Lunchtime? It . . .

Write the letters of each answer in the box containing the exercise number.

Complete the sentence.

A. When three lines, rays, or segments intersect in the same point, they are called
   ________ lines, rays, or segments.

B. The circumcenter of a triangle is ________ from the vertices of the triangle.

C. The angle ________ of a triangle are congruent.

D. The ________ of the triangle in the point of intersection of the angle bisectors.

Find the indicated measure using the diagram. The perpendicular bisectors are at points
D, E, and F. Angle bisectors are at A, B, and C.

6. Find GO.
   a. GO = 10, GO = 9, Find GO.
   b. GO = 8, GO = 5, Find GO.
   c. GO = 20, GO = 13, Find GO.
   d. GO = 12, GO = 9, Find GO.
   e. GO = 6, GO = 4, Find GO.
   f. GO = 10, GO = 7, Find GO.

Each Puzzle Time provides additional practice in a fun format in which
students use their mathematical knowledge to solve a riddle. This
format allows students to self-check their work. Your students can learn
the lesson concepts by finding the answers to silly jokes and riddles.

Cumulative Review

In Exercises 7–14, write the sentence as an inequality.

7. A number x is greater than or equal to 10.

8. A number x is more than 2 and less than 11.

9. A number x is at least 6 and less than 11.

10. A number x is less than or equal to 10.

11. A number x is greater than or equal to 10.

12. A number x is greater than 8 and less than 13.

13. A number x is fewer than 15 and at least 1.

14. A number x is less than 6 and greater than or equal to 10.

In Exercises 15–22, find the shape of a line passing through the given points.

15. (1, 6) and (8, 3)
16. (3, 4) and (10, 6)
17. (4, 6) and (11, 8)
18. (5, 7) and (12, 9)
19. (6, 8) and (13, 10)
20. (7, 9) and (14, 11)
21. (8, 10) and (15, 12)
22. (9, 11) and (16, 13)

In Exercises 23–30, write an equation of the line passing through points P and Q that is parallel
to the given line.

23. P(3, 4), Q(1, 2)
24. P(5, 6), Q(2, 4)
25. P(0, 5), Q(0, 2)
26. P(3, 4), Q(1, 2)
27. P(2, 3), Q(2, 1)
28. P(1, 2), Q(1, 0)
29. P(3, 4), Q(4, 5)
30. P(5, 6), Q(6, 7)
31. P(2, 3), Q(3, 4)
32. A local pizza shop charges $12.50 for a large, one-topping pizza. Each additional
topping is $2.50.
   a. Write an equation for the total cost C of a large pizza with toppings.
   b. How much does it cost for a cheese-topping pizza?
   c. How much does it cost for a two-topping pizza?
   d. How much does it cost for a five-topping pizza?
The **Student Journal** serves as a valuable component where students may work extra problems, take notes about new concepts and classroom lessons, and internalize new concepts by expressing their findings in their own words. *Available in English and Spanish*

## Maintaining Mathematical Proficiency

The Maintaining Mathematical Proficiency corresponds to the Pupil Edition Chapter Opener. Your students have the opportunity to practice prior skills necessary to move forward.

### Exercise 1

Write an equation of the line passing through point \( P \) that is perpendicular to the given line.

1. \((5, 2), y = 2 + 6x\)
2. \((6, 2), y = -3x - 3\)
3. \((-1, -2), y = -3x + 4\)
4. \((3, -1), y = -x - 1\)
5. \((0, 7), y = -x - 5\)
6. \((1, -4), y = x + 4\)

### Exercise 2

Write the sentence as an inequality.

7. A number \( g \) is at least 4 and no more than 12.
8. A number \( r \) is more than 2 and less than 7.
9. A number \( q \) is less than or equal to 6 or greater than 1.
10. A number \( p \) is fewer than 17 or no less than 5.
11. A number \( k \) is greater than or equal to –4 and less than 1.

## Exploration Journal

The Exploration pages correspond to the Explorations and accompanying exercises in the Pupil Edition. Your students can use the room given on these pages to show their work and record their answers.

### Essential Question

How are the sides related to the angles of a triangle? How are any two sides of a triangle related to the third side?

Go to [BigIdeasMath.com](https://www.bigideasmath.com) for an interactive tool to investigate this exploration.

Work with a partner. Use dynamic geometry software. Drag any vertex of \( \triangle ABC \).

### Exploration: Comparing Angle Measures and Side Lengths

Sample Points

- \( A(1, 3) \)
- \( B(5, 1) \)
- \( C(7, 4) \)

### Part a

Find the side lengths and angle measures of the triangle.

### Part b

Order the side lengths. Order the angle measures. What do you observe?

### Part c

Drag the vertices of \( \triangle ABC \). Record the side lengths and angle measures in the following table. Write a conjecture about your findings.

<table>
<thead>
<tr>
<th>( AB )</th>
<th>( BC )</th>
<th>( AC )</th>
<th>( \angle A )</th>
<th>( \angle B )</th>
<th>( \angle C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Notetaking with Vocabulary

This student-friendly notetaking component is designed to be a reference for key vocabulary, properties, and core concepts from the lesson. Students can write the definitions of the vocabulary terms in their own words and take notes about the core concepts.

Core Concepts

How to Write an Indirect Proof (Proof by Contradiction)

Step 1: Identify the statement you want to prove. Assume temporarily that this statement is false by assuming that its opposite is true.

Step 2: Reason logically until you reach a contradiction.

Step 3: Point out that the desired conclusion must be true because the contradiction proves the temporary assumption false.

Notes:

Theorems

Triangle Longer Side Theorem

If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

Notes:

Extra Practice

In Exercises 1–3, write the first step in an indirect proof of the statement.

1. Not all the students in a given class can be above average.
2. No number equals another number divided by zero.
3. The square root of 2 is not equal to the quotient of any two integers.

In Exercises 4 and 5, determine which two statements contradict each other. Explain your reasoning.

4. A. \( \triangle ABC \) is equilateral.  
   B. \( \triangle ABC \) is a right triangle.  
   C. \( \angle A \) is acute.  
   D. \( \angle C \) is obtuse.

5. A. \( \triangle ABC \) is a right triangle.  
   B. \( \angle A \) is acute.  
   C. \( \angle C \) is obtuse.

In Exercises 6–8, list the angles of the given triangle from smallest to largest.

In Exercises 9–12, is it possible to construct a triangle with the given side lengths? If not, explain why not.

9. 5, 12, 17  
10. 5, 31, 16  
11. 8, 5, 7  
12. 10, 3, 11

13. A triangle has two sides with lengths 5 inches and 13 inches. Describe the possible lengths of the third side of the triangle.
## Differentiating the Lesson

The **Differentiating the Lesson** online ancillary provides complete teaching notes and worksheets that address the needs of diverse learners. Lessons engage students in activities that often incorporate visual and kinesthetic learning. Some lessons present an alternative approach to teaching the content, while other lessons extend the concepts of the text in a challenging way for advanced students. Each chapter also begins with an overview of the differentiated lessons in the chapter and describes the students who would most benefit from the approach used in each lesson.

### Exploring Perpendicular Bisectors of Triangles

#### Lesson Preparation

**Materials:** Resource Sheet 3A: Triangle Trio Cards, tracing paper (or wax paper/patty paper), rulers, protractors, compasses

*For Optional Activity:

**Pre-assessment:** Pre-assess students’ understanding of how to construct a perpendicular bisector using tracing paper. Make copies of Resource Sheet 3A and cut apart the cards (one card per student). **Note:** If the number of students in the class is not divisible by three, allow some groups to have two of one type of triangle. Arrange desks into groups of three.

**Classroom Management:** Groups will be randomly generated through the Triangle Trio activity. Differentiate the groups by passing out the same card to students that need to be in different groups. Consider providing each group with a supply bin to store tools for the lesson.

#### Lesson Procedure

Distribute one Triangle Trio Card to each student before the start of the lesson and have protractors available for each student to help classify the given triangle. Explain to students that the cards they were given will determine their Triangle Trios (or groups of three students). Direct students to create their Triangle Trios by finding members with different types of triangles. Each group will contain an obtuse, acute, and right triangle.

Explain to students that they will be working with their Triangle Trios to develop conjectures about the perpendicular bisectors of a triangle. Provide each group with tracing paper and rulers and ask each student to use a ruler to copy his/her given triangle onto the tracing paper. **Note:** Students can create their own acute, obtuse, and right triangles, but ensure each group is testing at least one acute, one obtuse, and one right triangle. Explain to students that the more variety of triangles used, the more cases they will be able to test.
Credits

Cover
© Abidal | Dreamstime.com, Mopic/Shutterstock.com

Front Matter
v1 Goodluz/Shutterstock.com

Chapter 6
332 top left Warren Goldswain/Shutterstock.com; top right puttsk/
Shutterstock.com; center left sculpies/Shutterstock.com; bottom right
Olena Mykhaylova/Shutterstock.com; bottom left Nikonaf/Shutterstock.com;
345 Nikonaf/Shutterstock.com; 347 sababa66/Shutterstock.com;
349 Tobias W/Shutterstock.com; 353 filip robert/Shutterstock.com;
Shutterstock.com, John T Takai/Shutterstock.com; 358 top right Matthew
Cole/Shutterstock.com; center right artkamalov/Shutterstock.com,
ants on/Shutterstock.com; 360 center left Olena Mykhaylova/
Shutterstock.com; top right Lonely/Shutterstock.com; 367 Laschon
Maximilian/Shutterstock.com; 369 Nattika/Shutterstock.com;
373 sculpies/Shutterstock.com; 374 Shumo4ku/Shutterstock.com;
375 Alexandra Lande/Shutterstock.com, Dan Kosmayer/Shutterstock.com;
376 clearviewstock/Shutterstock.com; 380 Dzm1try/Shutterstock.com;
386 ©iStockphoto.com/enchanted_glass; 388 Warren Goldswain/
Shutterstock.com; 391 tuulijumala/Shutterstock.com, Brandon Bourdages/
Shutterstock.com